SYMMETRIES OF QUADRATIC FORMS CLASSES AND OF QUADRATIC SURDS CONTINUED FRACTIONS. PART I: A POINCARÉ MODEL FOR THE DE SITTER WORLD

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ABSTRACT. The problem of the classification of the indefinite binary quadratic forms with integer coefficients is solved introducing a special partition of the de Sitter world, where the coefficients of the forms lie, into separate domains. Every class of indefinite forms, under the action of the special linear group acting on the integer plane lattice, has a finite and well defined number of representatives inside each one of such domains. This property belongs exclusively to rational points on the one-sheeted hyperboloid.

In the second part we will show how to obtain the symmetry type of a class as well as its number of points in all domains from a sole representative of that class.

INTRODUCTION

In this paper by *form* we mean a binary quadratic form:

$$f = mx^2 + ny^2 + kxy \tag{1}$$

where m, n and k are integers and (x, y) runs on the integers plane lattice.

The integer number

$$\Delta = k^2 - 4mn$$

is called the *discriminant* of the form (1).

Following [1], we call a form *elliptic* if $\Delta < 0$, *hyperbolic* if $\Delta > 0$ and *parabolic* if $\Delta = 0$.

According to the usual terminology, the elliptic forms are said *definite* and, and *indefinite* the hyperbolic ones.

The problem of classifying and counting the orbits of binary quadratic forms under the action of $\mathrm{SL}(2, \mathbb{Z})$ on the (x, y)-plane dates back to Gauss and Lagrange ([5],[6]) and was recently re-proposed by Arnold in [1].

The description of the orbits of the positive definite forms by means of the action of the modular group on the Poincaré model of the Lobachevsky disc is well known: in this model, there is a special tiling of the disc such that every tile is in one-to-one correspondence with an element of the group, namely, the element that sends the fundamental domain to it. The upper sheet of the two-sheeted hyperboloid where the coefficients of

positive definite forms lie is represented by the Lobachevsky disc, so that every class of forms has one and only one representative in each domain.

The complement to the plane of the Lobachevsky disc, representing the hyperbolic forms, is not tiled by the same net of lines (for instance, the straight lines of the Klein model, separating the domains of the Lobachevsky disc) into domains of finite area.

In this article I show, however, that it is possible to introduce a special partition into separate domains of the one sheeted hyperboloid where the coefficients of the forms lie: in each of such domains every orbit has a finite – well defined – number of points[1].

The situation is, however, intrinsically different from that of the Lobachevsky disc, where there is no distinguished fundamental domain, i.e., all domains of the partition are equivalent. In our partition of the de Sitter world *there are two special domains*, that we call *fundamental*. An $\mathrm{SL}(2,\mathbb{Z})$ change of the system of coordinates (in the plane (x, y) of the forms, and, consequently, on the hyperboloid) changes the shape of *a finite subset of the partition's tiles* (including the shape of the fundamental domains), but preserves all the peculiar properties of the tiling:

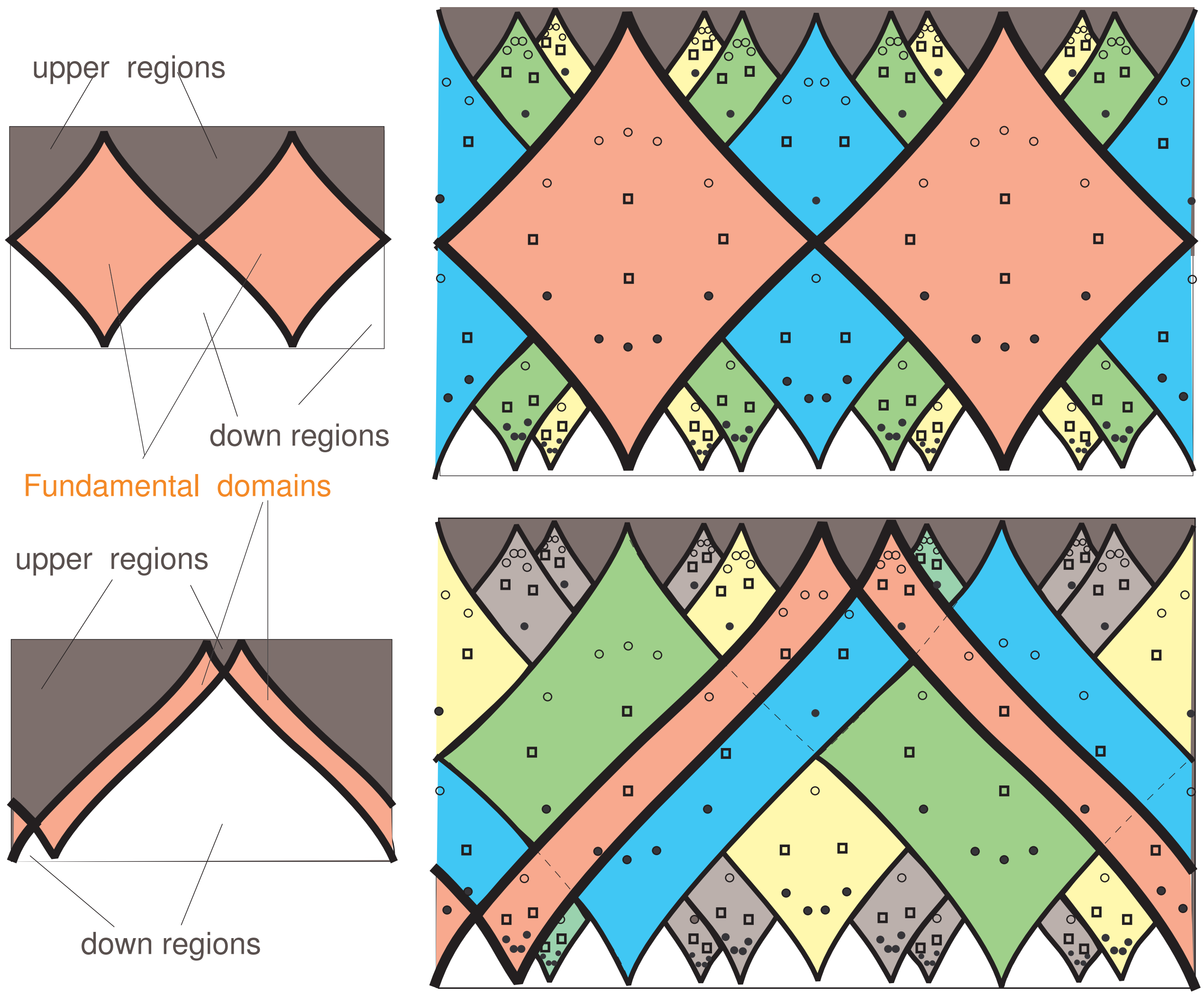
1) The complement to the fundamental domains of the hyperboloid is separated, by the fundamental domains, into four regions, two of which (called *upper regions*) are bounded from the circle at $+\infty$ of the hyperboloid and the other two regions (*down regions*) are bounded from the circle at $-\infty$ (note that the circles at infinite are invariant under the action of the group).

2) Each one of the two upper regions and of the down regions are partitioned into a countable set of domains which are in one-to-one correspondence with all elements of the semigroup of $\mathrm{SL}(2,\mathbb{Z})$ generated by $A = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$.

3) Every orbit has the same finite number of points, say N_u, in each domain of the upper regions, and the same finite number of points, say N_d, in each domain of the down regions. The fundamental domains contain $N = N_u + N_d$ of that orbit.

To understand this unusual situation (where the partition changes without changing the number of integer points in the corresponding domains) we give an example where the reader may verify the properties 1 and 3 above (see also Figure 14 to see more points of the orbits, and more domains).

[1]This is a surprising fact. Indeed, the orbit of a generic point (i.e., with irrational coordinates) on the de Sitter world is dense, as Arnold proved [2] (our results imply only that in each domain the number of points of such an orbit is unbounded).



This figure shows some tiles of two different partitions (related by a change of coordinates, namely by the operator BA) of the hyperboloid (projected onto an open cylinder) and the points of the three different classes of integer quadratic forms with $k^2 - 4mn = 32$, lying in these tiles.

The fundamental domains are marked by thick black boundary: they contain 5 points of the first orbit (circles), 5 points of the second orbit (black discs) and 4 points of the third orbit (squares).

Each domain in the upper regions contains 4 points of the first orbit, one point of the second, and 2 points of the third orbit. Each domain in the down regions contains 1 point of the first orbit, 4 points of the second, and 2 points of the third orbit.

The classical reduction theory introduced by Lagrange for the indefinite forms says that there is a finite number of forms such that m and n are positive and $m + n$ is less than k.

The reduction procedure, allowing to find these forms, can be described in terms of the model introduced in this work. We will see this relation in more detail in Part II.

The reduction theory that follows directly from our model is in fact closer to that expounded in the book ”The sensual (quadratic) forms” by J.H. Conway [4], since here the ‘reduced’ forms are those having $mn < 0$. We prefer this definition for the following reason: the number of reduced forms by Lagrange is equal to the number n_u of forms in each domain of the upper regions in our partition, whereas the number of reduced forms in our definition is the number $n_u + n_d$ of reduced forms in the fundamental domain.

However, we point out that the essential new element, with respect to the known theories, is the geometrical view-point, allowing to see the action of the group in the space of forms, exactly as for for the modular group action on the Lobachevsky disc.

We introduce here also the classification of the types of symmetries of the classes of forms. This classification is closely related to the classification of the symmetries of the quadratic surds continued fractions periods, answering more recent questions by Arnold [3], as we will show in the second part of the article.

We will see there also how to calculate the number of points in each domain for every class of hyperbolic forms from the coefficients of a form belonging to that class.

I am deeply grateful to Arnold who posed the problem in [1].

1. The space of forms and their classes

We will use also the following coordinates in the space of form coefficients:

$$\begin{aligned} K &= k, \\ D &= m - n, \\ S &= m + n. \end{aligned}$$

Remark 1. A point with integers coordinates (K, D, S) represents a form if and only if $D = S \mod 2$.

Remark 2. The discriminant in the new coordinates reads

$$\Delta = K^2 + D^2 - S^2.$$

Definition. A point with integer coordinates (m, n, k) or with integer coordinates (K, D, S) such that $D = S \mod 2$ is called a *good point*. It will be indicated by a bold letter.

1.1. Action of $\mathbf{SL}(2, \mathbb{Z})$ on the form coefficients.

Let $\mathbf{f}$ be the triple (m,n,k) of the coefficients of the form (1), and $\mathbf{f}'$ the triple (m',n',k'), corresponding to the form f' obtained from f by the action of an operator L of $\mathrm{SL}(2,\mathbb{Z})$. I.e., if $\mathbf{v}=(x,y)$, we define $f'(\mathbf{v})=f(L(\mathbf{v}))$.

We thus associate to L the operator T_L acting on $\mathbb{Z}^3$ in this way:

$$\mathbf{f}'=T_L\mathbf{f}. \tag{2}$$

This defines an homomorphism from $\mathrm{SL}(2,\mathbb{Z})$ to $\mathrm{SL}(3,\mathbb{Z})$: $L\mapsto T_L$. We denote by $\mathcal{T}$ the image of this homomorphism. The subgroup $\mathcal{T}$ is isomorphic to $\mathrm{PSL}(2,\mathbb{Z})$, since $T_L=T_{-L}$.

Definition. The *orbit*[2] of a good point $\mathbf{f}$ is the set of points obtained applying to $\mathbf{f}$ all elements of the group $\mathcal{T}$. The class of $\mathbf{f}=(m,n,k)$ is denoted by $C(\mathbf{f})$ or by $C(m,n,k)$.

The following statements are obvious or easy to prove:

– All points of an orbit are good.

– All points of an orbit belong to the hyperboloid $K^2+D^2-S^2=\Delta$. Moreover, in the elliptic case, the orbit lies entirely either on the upper or on the lower sheet of the hyperboloid; in the parabolic case, it lies entirely either on the upper or on the lower cone.

– Every good point belongs to one orbit.

– Different orbits are disjoint.

1.2. **The subgroups $\mathcal{T}^+$ and $\mathcal{T}^-$.**

Consider the following generators of the group $\mathrm{SL}(2,\mathbb{Z})$:

$$\mathbf{A}=\begin{pmatrix}1&1\\0&1\end{pmatrix},\quad \mathbf{B}=\begin{pmatrix}1&0\\1&1\end{pmatrix},\quad \mathbf{R}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}. \tag{3}$$

and their inverse operators denoted by $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$ and $\bar{\mathbf{R}}$.

Note that

$$\mathbf{R}=\bar{\mathbf{B}}\mathbf{A}\bar{\mathbf{B}}=\mathbf{A}\bar{\mathbf{B}}\mathbf{A}\quad\text{and}\quad \bar{\mathbf{R}}=\bar{\mathbf{A}}\mathbf{B}\bar{\mathbf{A}}=\mathbf{B}\bar{\mathbf{A}}\mathbf{B}. \tag{4}$$

We denote the corresponding operators $T_{\mathbf{A}}$, $T_{\mathbf{B}}$, $T_{\mathbf{R}}$ of $\mathcal{T}$, obtained by eq. (2), by A, B, R and their inverse by $\bar{A}$, $\bar{B}$ and $\bar{R}$.

Remark. The matrices of A and B are one the transpose of the other[3] as well as those of $\mathbf{A}$ and $\mathbf{B}$, whereas the transpose of $\mathbf{R}$ is equal to $\mathbf{R}^{-1}$. Since the transpose of R is equal

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[2] Words "orbit" and "class" are synonymous.

[3] The matrices of the generators of $\mathcal{T}$, A, B, and R are, in coordinates (m,n,k):

$$A=\begin{pmatrix}1&0&0\\1&1&1\\2&0&1\end{pmatrix},\quad B=\begin{pmatrix}1&1&1\\0&1&0\\0&2&1\end{pmatrix},\quad R=\begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}.$$

to $\bar{R} = R$, relations (4) become:

$$R = \bar{B}A\bar{B} = A\bar{B}A = \bar{A}B\bar{A} = B\bar{A}B. \tag{5}$$

Definition. We call $\mathcal{T}^+$ ($\mathcal{T}^-$) the multiplicative semigroup of the elements of $\mathcal{T}$ generated the identity and by the operators A and B ($\bar{A}$ and $\bar{B}$).

Lemma 1.1. *a) Every operator $T \in \mathcal{T}^+$ ($T \in \mathcal{T}^-$) is written in a unique way as a product of its generators. b) Every operator $T \in \mathcal{T}$ can be written as VSU, where S belongs to $\mathcal{T}^+$ and the operators U and V are equal either to the identity operator or to R. Statement (b) holds as well replacing $\mathcal{T}^+$ by $\mathcal{T}^-$.*

Proof. a) There are no relations involving only operators $\mathbf{A}$ and $\mathbf{B}$ in $\mathrm{SL}(2,\mathbb{Z})$, hence we have no relations involving only A and B. b) Relations (5) allow to transform any word in A, B, R and their inverse operators into a word of type VSU. This lemma is illustrated by Figure 3; indeed, there is an one-to-one correspondence between the elements of the group and the domains of the Lobachevsky disc. The element of the group corresponding to a domain indicates that the fundamental domain (I) is sent to this domain by that element. One sees that any domain in the right half-disc is attained, from the fundamental domain, by an operator which can be written as an element of $\mathcal{T}^+$ followed by the identity or by R, and any domain in the left half-disc by an element of $\mathcal{T}^-$ followed by the identity or by R. The multiplication by R at left acts as a reflection with respect to the centre. Hence every domain in the right half-disc can be attained by the operator corresponding to the domain symmetric to it with respect to the centre, multiplied at left by R, and vice versa. Hence any domain can be written using an element of $\mathcal{T}^+$ (as well as $\mathcal{T}^-$), multiplied at left and at right by the identity of by R. $\square$

1.3. **Symmetries of the form classes.**

We introduce in this section some different types of symmetries that the classes of forms may possess.

We define each one of these symmetries as the invariance of the class under the reflection with respect to some plane or some axis through the center of the coordinate system, plane or axis which is *not invariant* under the action of the group $\mathcal{T}$. So, a priori these symmetries could hold no more in another system of coordinates. However, we prove that the action of the group $\mathcal{T}$ *preserves* each one of the symmetries, and hence the

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The matrices of the same generators in coordinates (K, D, S) are

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1/2 & -1/2 \\ 1 & 1/2 & 3/2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1/2 & 1/2 \\ 1 & -1/2 & 3/2 \end{pmatrix}, \quad R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

same definitions of the symmetries *hold in any system of coordinates obtained by a $\mathcal{T}$ coordinate transformation.* This is equivalent to say that a symmetry of a class of forms is the symmetry by respect to all infinite planes (or axes), which are the images under $\mathcal{T}$ of one of such symmetry planes (or axes).

To every form $\mathbf{f} = (m, n, k)$ there correspond 8 forms, obtained by 3 involutions (see Figure 1):

$$\mathbf{f}_c = (n, m, -k), \quad \overline{\mathbf{f}} = (m, n, -k), \quad \mathbf{f}^* = (-n, -m, k). \tag{6}$$

All these involutions commute, since correspond to changes of sign of some of the coordinates K, D, S. In these coordinates,

$$\mathbf{f}_c = (-K, -D, S), \quad \overline{\mathbf{f}} = (-K, D, S), \quad \mathbf{f}^* = (K, D, -S).$$

The 8 forms defined by these involution on the form $\mathbf{f}$ lie on the same hyperboloid as $\mathbf{f}$.

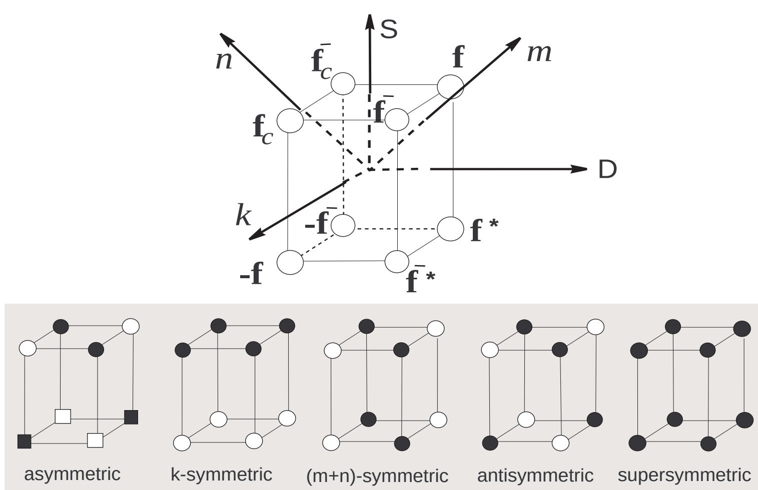


FIGURE 1. For every symmetry type, the forms denoted by the same symbol and the same color belong to the same class.

The *complementary* form $\mathbf{f}_c$ always belongs to the class of $\mathbf{f}$, since $\mathbf{f}_c = R\mathbf{f}$, and $R \in \mathcal{T}$. Note that the complementary form $\mathbf{f}_c$ of the form $\mathbf{f}$ satisfies in the (x, y) plane $f_c(x, y) = f(y, -x) = f(-y, x)$ and the corresponding PSL$(2, \mathbb{Z})$-change of coordinates is a rotation by $\pi/2$.

The complementary of the *conjugate* form $\overline{\mathbf{f}}_c = (n, n, k)$ of the form $\mathbf{f} = (m, n, k)$ corresponds to the reflection of the plane (x, y) with respect to the diagonal (whose operator $\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$) does not belong to SL$(2, \mathbb{Z})$).

Note that the opposite form $-\mathbf{f} = (-m, -n, -k) = (-K, -D, -S)$ is the complementary of the adjoint of $\mathbf{f} = (m, n, k) = (K, D, S)$, $(-\mathbf{f} = \mathbf{f}_c^*)$.

The forms obtained from a form $\mathbf{f}$ by the conjugation and/or adjunction may or may not belong to $C(\mathbf{f})$. However, *if a class contains a pair of forms related by some involution, or a form which is invariant under some involution, then the entire class is invariant under such involution*:

Proposition 1.2. *Let σ be one of the following involutions: $\sigma(\mathbf{f}) = \bar{\mathbf{f}}$ or $\sigma(\mathbf{f}) = \mathbf{f}^*$ or $\sigma(\mathbf{f}) = \overline{\mathbf{f}}^*$ Suppose that, for some $\mathbf{f}$, $\sigma(\mathbf{f}) \in C(\mathbf{f})$. Then $\sigma(\mathbf{g}) \in C(\mathbf{f})$ for all $\mathbf{g} \in C(\mathbf{f})$.*

Proof. We have to prove the following lemma.

Lemma 1.3. *The following identities hold:*

$$\begin{array}{lll} 1) & (A\mathbf{f})^* = \bar{B}\mathbf{f}^*; & (B\mathbf{f})^* = \bar{A}\mathbf{f}^*; \\ 2) & \overline{A\mathbf{f}} = \bar{A}\ \bar{\mathbf{f}}; & \overline{B\mathbf{f}} = \bar{B}\ \bar{\mathbf{f}}; \\ 3) & (\overline{A\mathbf{f}})^* = B\overline{\mathbf{f}}^*; & (\overline{B\mathbf{f}})^* = A\overline{\mathbf{f}}^*. \end{array} \tag{7}$$

Proof of the lemma.

Let $\mathbf{f} = (m, n, k)$. We have $A\mathbf{f} = (m, m+n+k, k+2m)$, $B\mathbf{f} = (m+n+k, n, k+2n)$.

1) Since $\mathbf{f}^* = (-n, -m, k)$, we have

$(A\mathbf{f})^* = (-m-n-k, -m, k+2m)$ and $\bar{B}\mathbf{f}^* = (-n-m-k, -m, k+2m)$;

$(B\mathbf{f})^* = (-n, -n-m-k, k+2n)$ and $\bar{A}(\mathbf{f}^*) = (-n, -m-n-k, k+2n)$.

2) Since $\overline{\mathbf{f}} = (m, n, -k)$, we have

$\overline{A\mathbf{f}} = (m, m+n+k, -k-2m)$ and $\bar{A}\ \overline{\mathbf{f}} = (m, m+n+k, -k-2m)$;

$\overline{B\mathbf{f}} = (m+n+k, n, -k-2m)$ and $\bar{B}\ \overline{\mathbf{f}} = (m, m+n+k, -k-2m)$.

3) Since $\overline{\mathbf{f}}^* = (-n, -m, -k)$, we have

$\overline{A\mathbf{f}} = (m, m+n+k, -k-2m)$, $(\overline{A\mathbf{f}})^* = (-n-m-k, -m, -k-2m)$ and $B\overline{\mathbf{f}}^* = (-n-m-k, -m, -k-2m)$;

$\overline{B\mathbf{f}} = (m+n+k, n, -k-2n)$, $(\overline{B\mathbf{f}})^* = (-n, -n-m-k, -k-2n)$ and $A\overline{\mathbf{f}}^* = (-n, -m-n-k, -k-2n)$. □

We observe now that every operator $T \in \mathfrak{T}$ can be written as product of the generators A, B and their inverse. Then the above lemma implies that $\sigma(T\mathbf{f}) = T'\sigma(\mathbf{f})$ for some $T' \in \mathfrak{T}$. Therefore, if $\sigma(\mathbf{f}) \in C(\mathbf{f})$, then, for every $T \in \mathfrak{T}$, $\mathbf{g} = T\mathbf{f} \in C(\mathbf{f})$ and hence $\sigma\mathbf{g} \in C(\mathbf{f})$. □

By the above Proposition, the following definitions hold in all coordinate systems (equivalent under $\mathfrak{T}$-transformations).

Definition.

A class of forms is said to be (see Fig. 1)

(1) *asymmetric*, if it is only invariant under reflection with respect to the axis of the coordinate S ($K=0$, $D=0$), so containing only pairs of complementary forms;

(2) k-*symmetric*, if it is not supersymmetric but it is invariant under reflection with respect to the plane $k=0$ ($K=0$). It contains, with every $\mathbf{f}$, its *conjugate* form $\overline{\mathbf{f}}$;

(3) $(m+n)$-*symmetric*, if it is not supersymmetric but it is invariant under reflection with respect to the plane $m+n=0$ ($S=0$). It contains, with every $\mathbf{f}$, its *adjoint* form $\mathbf{f}^*$;

(4) *antisymmetric*, if it is not supersymmetric but it is invariant under reflection with respect to the planes $m=0$ and $n=0$ ($|S|+|D|=0$). It contains, with every $\mathbf{f}$, its *antipodal* form (the conjugate of the adjoint) $\overline{\mathbf{f}}^* = (-n,-m,-k) = (-K,D,-S)$.

(5) *supersymmetric*, if it contains all 8 forms obtained by the 3 involutions;

Remarks. 1) Note that the opposite form $-\mathbf{f}$, being the the complementary of the adjoint of the form $\mathbf{f}$, belongs to the class of $\mathbf{f}$ only if the class is $(m+n)$-symmetric or supersymmetric (see Fig. 1).

2) A class of elliptic forms containing $\mathbf{f}$ cannot contain neither $-\mathbf{f}$, nor $\mathbf{f}^*$ nor $\overline{\mathbf{f}}^*$. Hence it is either asymmetric or k-symmetric.

2. Elliptic forms

In this section the classification of positive definite forms is treated in order to introduce some notions and terms which will be used in Sections 3 and 4.

We define a map from one sheet of the two-sheeted hyperboloid to the open unitary disc, which gives explicitly the one-to-one correspondence between the integer points of an orbit on the hyperboloid and the domains of the classical Poincaré model of the Lobachevsky disc.

Let $\mathcal{P}$ be the following *normalized* projection from the upper sheet of the hyperboloid $K^2+D^2-S^2=\Delta$ ($\Delta<0$) to the disc of unit radius. Let $\mathbf{p}=(K,D,S)$ be a point on the hyperboloid (see Figure 6, left), $\mathbf{p}'$ its projection to the disc of radius $\rho=\sqrt{-\Delta}$ from the point O', and $\mathcal{P}\mathbf{p}$ the image of the normalized projection. We have

$$\mathcal{P}\mathbf{p} = \begin{cases} \widetilde{K} = \frac{K}{\rho+S} \\ \widetilde{D} = \frac{D}{\rho+S}. \end{cases} \tag{8}$$

Let $L=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be any operator of $\mathrm{SL}(2,\mathbb{Z})$ and T_L its corresponding operator defined by (2).

Definition. We define the operator $\widetilde{L}$ acting on the disc of radius 1 by:

$$\widetilde{L}(\mathcal{P}\mathbf{p}) = \mathcal{P}(T_L\mathbf{p}). \tag{9}$$

On the other hand, the operator $L \in \mathrm{SL}(2,\mathbb{Z})$ defines another map from the disc to itself. Indeed, let H_L be the homographic operator acting on the upper complex half-plane $\{z \in \mathbb{C} : \Im(z) \geq 0\}$:

$$H_L z = \frac{az+b}{cz+d}. \tag{10}$$

The following map $\pi : z \to w$ sends the upper complex half plane to the unitary complex disc $\{w \in \mathbb{C} : |z| \leq 1\}$ (Figure 2)

$$w = \pi z = \frac{1+iz}{1-iz}. \tag{11}$$

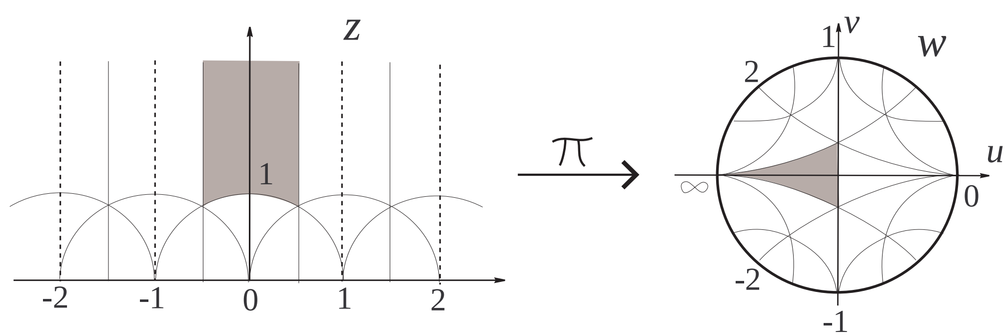


Figure 2. The fundamental domain is gray

Define the operator $\widehat{L}$ acting on the complex unitary disc by:

$$\widehat{L}(\pi z) = \pi(H_L z). \tag{12}$$

Proposition 2.1. *The actions of operators $\widehat{L}$ and $\widetilde{L}$ on the unitary disc do coincide under the identification:*

$$\widetilde{D} = \Re w, \qquad \widetilde{K} = \Im w.$$

The proof is done by writing explicitly the operators corresponding to the $\mathrm{SL}(2,\mathbb{Z})$ generators and by comparing their actions in coordinates. $\square$

Remark. The group of operators defined by eq. (9) is isomorphic to the homographic group of the operators H_L, and to the group $\mathcal{T}$, i.e., to $\mathrm{PSL}(2,\mathbb{Z})$: we will denote its generators by the same letters as the corresponding generators of $\mathcal{T}$.

We take as coordinates in the Lobachevsky disc the pair $(\widetilde{K}, \widetilde{D})$. Hence our Lobachevsky disc is obtained from the unitary complex disc with coordinate $w = u + iv$ (see Figures 2 and 3) by the reflection with respect to the diagonal $v = u$.

In Figure 3 the Lobachevsky disc is shown with a finite sets of domains. The fundamental domain is indicated by the letter I (the bold line at the frontier belong to it, the dotted line does not). The other domains are attained applying to I the elements of $\mathrm{PSL}(2, \mathbb{Z})$, written in terms of R, A, B and their inverse.

Because of relations (5) involving these generators, the expressions are not unique. We have chosen this representation in order to see the meaning of Lemma 1.1, which will be determinant in Section 4.

Remark. The choice of the fundamental domain is arbitrary. Every domain has the shape of a triangle with one and only one corner on the circle at infinity. Choose an arbitrary triangle, labeled by the operator M, and orient it as the fundamental one, with the corner on the circle at bottom. Then the triangle adjacent at right is labeled by MA, the triangle adjacent at left by $M\bar{A}$, and the triangle adjacent at the base by MR, and so on. Replacing M by I in all triangles, the arbitrarily chosen triangle becomes the new fundamental domain.

Every orbit has one and only one point in every domain. In Figure 4 the Lobachevsky disc with a finite subset of domains is shown together with a finite part of the three distinct orbits in the case $\Delta = -31$.

Remark. The opposite, the antipodal, and the adjoint of a positive (negative) definite quadratic form f are negative (resp., positive) definite quadratic forms, so they cannot belong to the same class of f. For this reason, a class of elliptic form may have only two types of symmetries: it is either k-symmetric or asymmetric.

2.1. **The hierarchy of the points at the infinity.**

This section is important for the study of hyperbolic forms.

The points of the circle C (the circle at infinite bounding the Lobachevsky disc) having rational coordinates, inherit, by the $\mathrm{PSL}(2, \mathbb{Z})$ action, a hierarchy, that we will recall. These points will be thus distinguished into *points of zero, first, second (and so on) generation*. We notice that this hierarchy will be the starting point of our partition of the de Sitter world.

Denote by π' the composition of mapping π (see eq. (11)) with the reflection of the disc-image ($|w| \leq 1$) with respect to the diagonal ($\Im w = \Re w$).

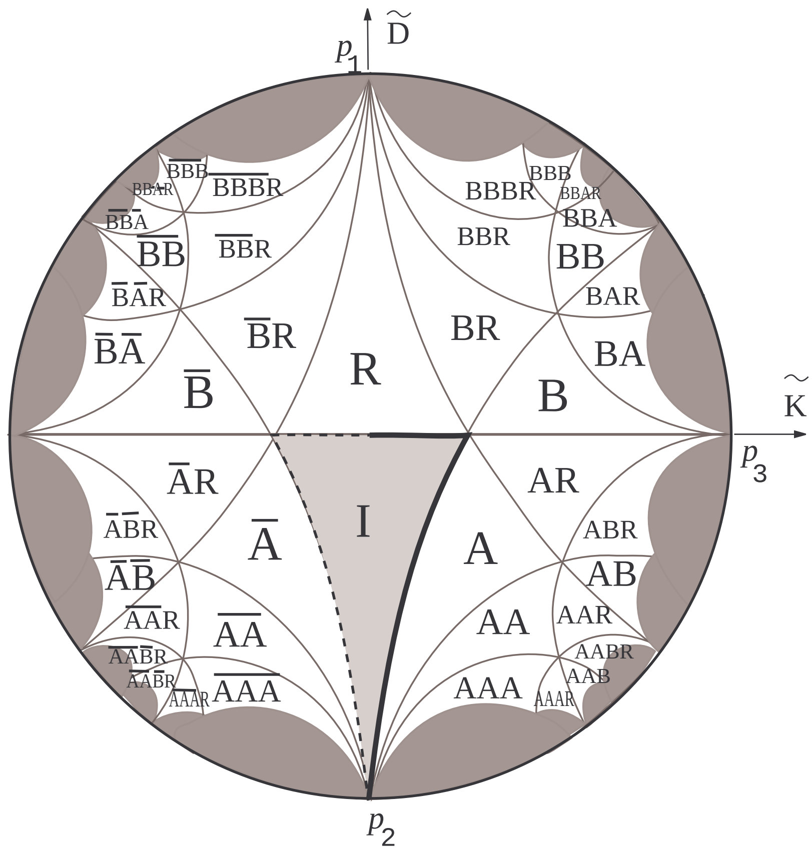


FIGURE 3. A finite set of domains in the Lobachevsky disc with coordinates $\widetilde{K}, \widetilde{D}$.

We consider, because of the symmetry of the picture, only the right part of the circle C. This semicircle ($\widetilde{K} \geq 0$) is the image by π' of the half real line $x \equiv \Re z > 0$ of the half-plane where the homographic operators act.

The points $p_1 = (\widetilde{K} = 0, \widetilde{D} = 1)$ and $p_2 = (\widetilde{K} = 0, \widetilde{D} = -1)$ of zero generation (i.e., the end-points of this semicircle) are the images by π' of the points $x_1 = 0$ and $x_2 = \infty$ of the real half-line.

The rational points x_i of the half real line are written as fractions, i.e.: $0 \equiv 0/1$, $\infty \equiv 1/0$, $q \equiv q/1$, if $q \in \mathbb{Z}$ etc, and points p_i are their images by π' on the circle C.

Here A and B are the generators of the homographic group that correspond to the generators $\mathbf{A}$ and $\mathbf{B}$ of $\mathrm{SL}(2, \mathbb{Z})$.

$$\mathrm{A} : x \to \frac{x+1}{1}; \quad \mathrm{B} : x \to \frac{x}{x+1}. \tag{13}$$

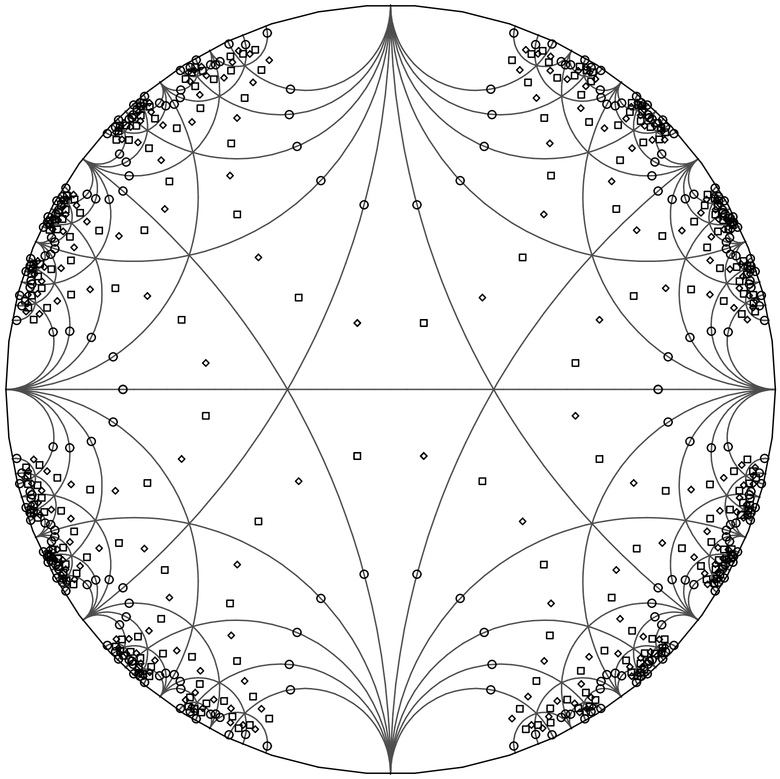


FIGURE 4. Finite subsets of the 3 distinct orbits in the case $\Delta = -31$ (the representative point (m, n, k) is in the fundamental domain). Two asymmetric orbits: boxes (2,4,-1) and rhombi (2,4,1), and one k-symmetric: circles (1,8,1). Unitary disc coordinates are $(\widetilde{K}, \widetilde{D})$.

Consider the iterate action of such generators on the points $x_1 = 0$ and $x_2 = \infty$. We have firstly:

$$\mathrm{A}x_1 = x_3; \quad \mathrm{B}x_1 = x_1; \quad \mathrm{A}x_2 = x_2; \quad \mathrm{B}x_2 = x_3, \tag{14}$$

where $x_3 = 1/1$ is the preimage by π' of the point $p_3 = (\widetilde{K} = 1, \widetilde{D} = 0)$.

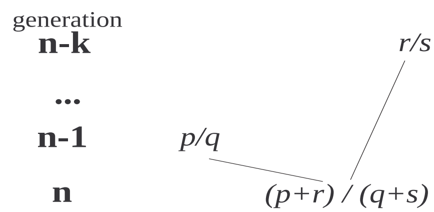
Definition. *The points x_i of the n-th generation are obtained from the point of first generation $x_2 = 1/1$ applying to it all the 2^n words in the generators* A *and* B*. The hierarchy and the order of these points is shown in the following scheme, where* T *indicates any word of $2^{(n-1)}$ generators:*

generation				
n		T(1/1)		
n+1	TB(1/1)		TA(1/1)	
n+2	TBB(1/1)	TBA(1/1)	TAB(1/1)	TAA(1/1)

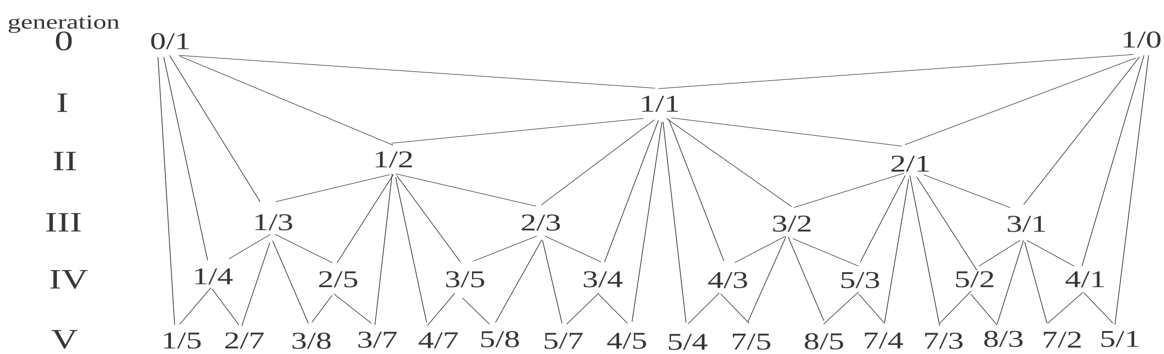
The points of all generations have a nice algebraic property that we recall.

Definition. We call *sons* of the point T(1/1), belonging to the n-th generation, the points TA(1/1) and TB(1/1) of the $(n+1)$-th generation. T(1/1) is thus the *father* of his sons. In the scheme above the segments indicate the relations father-son.

Farey rule. *The coordinate of a point of the n-th generation can be calculated directly from those of his father and of the nearest (i.e. the closest in the line where these points lie) ancestor to his father, by the rule shown in the following scheme:*



In the following scheme of the hierarchy every point x_i is connected by a segments to its 2 sons, to its father and to its closest ancestor. Note that the descendants of B(1/1) are the inverse fractions of the descendants of A(1/1).



Remark. By this procedure all positive rational numbers are covered.

The relations of order and the hierarchy are preserved by the map π'. Hence we have the same ordering and hierarchy on the rational points on the circle C.

Note that the point $(\widetilde{K} = -1, \widetilde{D} = 0)$ on the circle C is the image by π' of point $-1/1$: the above construction can be repeated by the iterate action of the inverse generators A^{-1} and B^{-1} on point $(-1/1)$, by considering the point $-1/0 = -\infty$ as the preimage of point p_2. In this way one finds the rational points in the half real line $x < 0$, and the ordering and the hierarchy of the points with rational coordinates of the left part of circle C.

The point $x_i = \frac{p}{q}$ is sent by π' to the point of C with coordinates $(\widetilde{K}, \widetilde{D})$:

$$\widetilde{K} = \frac{2pq}{p^2 + q^2}, \qquad \widetilde{D} = \frac{p^2 - q^2}{p^2 + q^2}. \tag{15}$$

Remark. The map π' defines a map from the set of rational numbers to the set of Pythagorean triples $\{(a,b,c)\in\mathbb{Z}^3:\ a^2+b^2=c^2\}$:

$$\frac{p}{q}\rightarrow\left(a=2pq,\quad b=q^2-p^2,\quad c=p^2+q^2\right).$$

3. Parabolic forms

Definition. A good point (K,D,S) is called *Pythagorean* if $K^2+D^2=S^2$. If K,D,S have no common divisors, then the triple is said *simple*. If (K,D,S) is Pythagorean, then the set of points $\{(\lambda K,\lambda D,\lambda S),\ \lambda\in\mathbb{Z}\}$, all Pythagorean, is called *Pythagorean line.*

Pythagorean points belong to the cone $\Delta=0$, and any good point belonging to the cone is Pythagorean.

Lemma 3.1. *There exists an one-to-one correspondence between the Pythagorean lines and the points with rational coordinates p_i on the circle C.*

Proof. By the last remark of the preceding section, to every point p_i on the circle C we associate a Pythagorean triple. This triple represents a good point because $b-c=0$ mod 2. On the other hand, having a simple good point (K,D,S), the equations

$$K=2pq,\quad D=p^2-q^2,\quad S=p^2+q^2$$

have solution $p=\sqrt{(S+D)/2}$; $q=\sqrt{(S-D)/2}$, i.e., $p=\sqrt{m}$, $q=\sqrt{n}$.

Since $S=m+n$ and $D=m-n$, and in this case $K=2\sqrt{mn}$, m and n have no common divisors, otherwise triple (K,D,S) should be not simple. But the equality $K^2=4mn$, with m and n relatively prime, implies that $m=p^2$ and $n=q^2$ for some integers p and q. Hence to every simple Pythagorean triple we associate a point p_i on the circle C and vice versa. The Pythagorean line corresponding to p_i is the line through 0 and p_i on the cone of parabolic forms. $\square$

Theorem 3.2. *There are infinitely many classes of forms with $\Delta=0$. In particular, all forms of type ax^2, ($m=a,n=0,k=0$) with $a\in\mathbb{Z}$, belong to different orbits, and every orbit on the cone contains a form of this type.*

Proof. We prove that for all $a\in\mathbb{Z}$ the orbits containing the points $(K=0,D=a,S=a)$ are distinct. Suppose that point $\mathbf{r}=(0,b,b)$ belong to the same orbit of point $\mathbf{p}=(0,a,a)$. Thus the form $f=ax^2$ is in the same class of the form $f'=bx^2$. This means that there exists an operator $L=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ of $\mathrm{SL}(2,\mathbb{Z})$ such that $a(\alpha x+\beta y)^2=bx^2$. This can be satisfied only by $\beta=0$, and, since $\alpha\delta-\gamma\beta=1$, by $\alpha=\delta=1$. Hence $a=b$. On the other hand, any parabolic form $f=mx^2+ny^2+kxy$, being $k^2-4mn=0$, can be written as

$a(\alpha x + \beta y)^2$, a being the greatest common divisor of (m, n, k). For every pair of integers α and β, there exist two integers γ and δ such that $\alpha\delta - \gamma\beta = 1$. So the inverse of the operator L is the operator of $\mathrm{SL}(2, Z)$ transforming form f into ax^2. □

4. Hyperbolic forms

Let X be the infinite set of planes through the origin in the 3-dimensional space (K, D, S), obtained from the plane $D = 0$ by the action of group $\mathcal{T}$. These planes subdivide the interior of the cone $(K^2 + D^2 < S^2)$ into domains. (Some of these planes are shown in Figure 5). These planes, intersecting both sheets of the two-sheeted hyperboloid, subdivides them into domains; those which belong to the upper sheet, are are projected by $\mathcal{P}$ (see Section 2) to the interior of domains of the Lobachevsky disc.

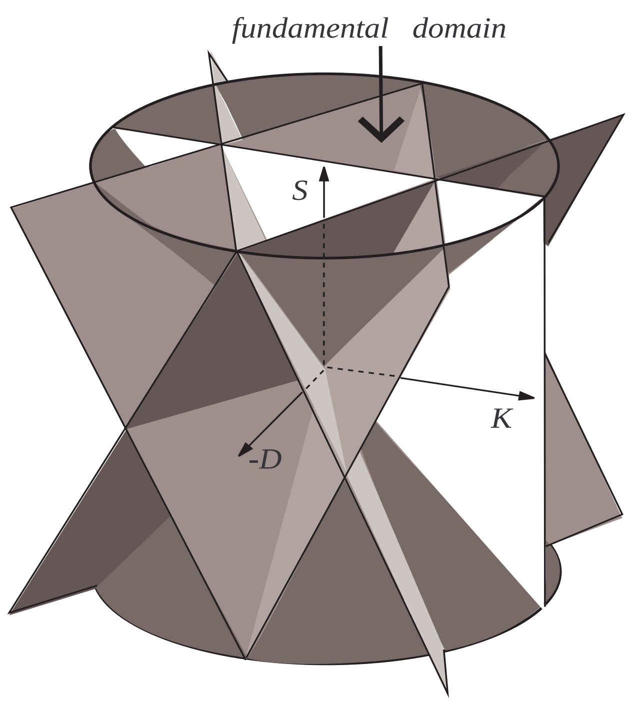


FIGURE 5. The fundamental domain in the space of forms coefficients.

The closure $\overline{X}$ of X contains also the planes tangent to the cone along all Pythagorean lines.

The intersection of the planes of $\overline{X}$ with the one-sheeted hyperboloid H $(K^2+D^2-S^2 = 1)$, at the exterior of the cone, forms a net of lines which is dense in H. For this reason there is no a natural prolongation of the Poincaré model of the Lobachevsky disc to the exterior of the disc - i.e., on the de Sitter world.

The horizontal section $S = 1$ of the set $\overline{X}$ gives, at the interior of the unit circle, the Klein model of the Lobachevsky disc. The arcs of circles between two points of the circle at infinite of the Poincaré model are substituted by the chords connecting these points. We are interested in the prolongations of these chords outside the disc: the description of the de Sitter world is based on a particular subset of these chords: they are the "limit" chords (the section $S = 1$ of the planes tangent to the cone) i.e., the tangents to the circle at all rational points of it, provided with the hierarchy explained in Section 2.1.

4.1. **The Poincaré model of the de Sitter world.**

In analogy with the standard projection $\mathcal{P}$ from the upper sheet of the two sheeted hyperboloid to the Lobachevsky disc, I have chosen the following mapping $\mathcal{Q}$ from the hyperboloid H, with equation $K^2 + D^2 - S^2 = \Delta$ in coordinates (K, D, S), to the open cylinder C_H:

$$C_H = \{(K, D, S) \ : \ K^2 + D^2 = 1, \ \ |S| < 1\}.$$

Definition. Let $\rho = \sqrt{\Delta}$ and $r = \sqrt{K^2 + D^2}$. The coordinates on the cylinder C_H are:

$$s = \frac{S}{r + \rho}, \tag{16}$$

and ϕ is the angle defined by the relations: $K = r \cos \phi$ and $D = r \sin \phi$ (see Figure 6, right).

The border of the cylinder consists of two circles, denoted c_1 ($s = 1$) and c_2 ($s = -1$).

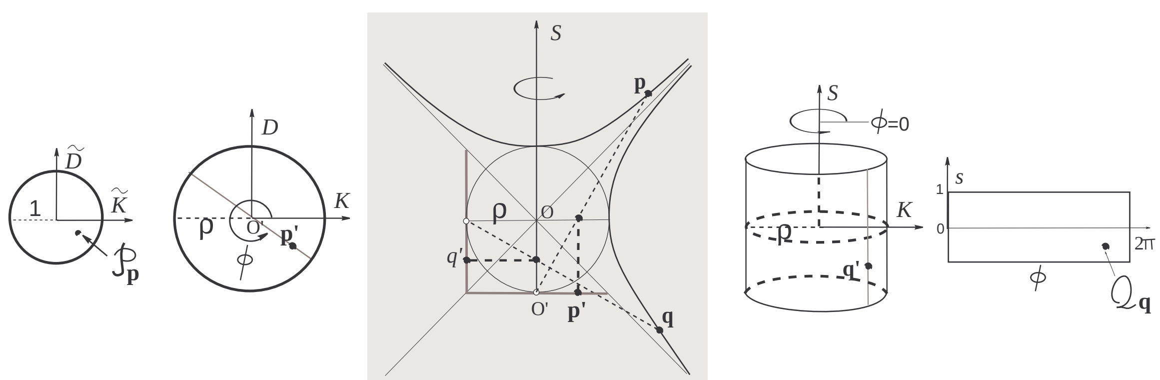


FIGURE 6. Projection $\mathcal{P}$, left, and $\mathcal{Q}$, right

Remark. The lines of intersection of the planes tangent to the cone with the hyperboloid H are straight lines, being generatrices of the hyperboloid.

Definition. We denote by H^0 and by H^0_R the domains

$$H^0 = \{(K, D, S) \in H : |S| < |D|, D > 0\};$$

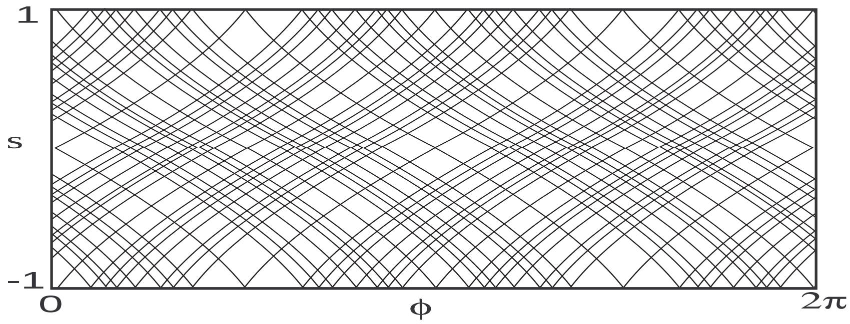


FIGURE 7. Projection on the cylinder C_H of few lines of the infinite set of lines, intersection of the planes tangent to the cone along Pythagorean lines.

$$H^0_R = \{(K, D, S) \in H : |S| < |D|, D < 0\}.$$

Remark. $H^0_R = RH^0$.

Since the mapping $\mathcal{Q}$ is one-to-one, for simplicity we denote by the same letters the domains on H and their images under $\mathcal{Q}$ on the cylinder C_H.

Circles c_1 and c_2 on the planes $S = 1$ and $S = -1$ coincide with the circle C at the infinite of the Lobachevsky disc. Hence also on c_1 and c_2 the points having rational coordinates $(\widetilde{K}, \widetilde{D})$ are mapped, according to (15), into the points p_i of first, second, third -and so on- generation with the hierarchy explained in Section 2.1.

In Figure 8, on the upper circle c_1, the points p_i up to the second generations are denoted by the corresponding rational numbers x_i.

Note that the upper (and lower) vertices of the domain H^0 and H^0_R, with coordinates $\phi = \pi/2$ and $\phi = 3\pi/2$, correspond to the points $x_1 = 0/1$ and $x_2 = 1/0 = \infty$.

Definition. We denote by H^{x_i} and H^{-1/x_i} the domains obtained by H^0 and H^0_R by a rigid translation to right, such that the the upper vertex of H^0 transfers to points x_i, and the upper vertex of H^0_R transfers to point $-1/x_i$. So, domains H^{x_i} inherit the hierarchy of points x_i: $H^0 = H^{0/1}$ and $H^0_R = H^{1/0}$ are called *rhombi of zero-generation*, $H^{1/1}$ and $H^{-1/1}$ *rhombi of first generation*, $H^{1/2}$, $H^{-2/1}$, $H^{-1/2}$, $H^{2/1}$ *rhombi of second generation* and so on.

Definition. We denote by H^O, H^I, H^{II}, H^{III} etc the unions of the rhombi of zero, first, second, third etc generations.

Definition. We call the *Poincaré model* of the hyperboloid H the cylinder C_H provided with the subdivision into domains obtained by the following procedure. Let $\mathcal{H}^0 = H^0 \cup H^0_R$ be the *domain of zero generation*. Let $\mathcal{H}^I = H^I \setminus (H^I \cap \overline{H^O})$ the domain of first generation, $\mathcal{H}^{II} = H^{II} \setminus (H^{II} \cap (\overline{H^O} \cup \overline{H^I}))$ the domain of second generation and so on (where $\overline{H^{\mathbf{n}}}$ is

the closure of $H^{\mathbf{n}}$): the domain of the n-th generation are thus obtained as

$$\mathcal{H}^{\mathbf{n}} = H^{\mathbf{n}} \setminus (H^{\mathbf{n}} \cap (\overline{H^{O}} \cup \overline{H^{I}} \cup \overline{H^{II}} \cup \cdots \cup \overline{H^{\mathbf{n-1}}})).$$

Figure 8 shows the domains of different generations. Note that the domain of n-th generation, $\mathcal{H}^{\mathbf{n}}$, has 2^{n+1} connected components.

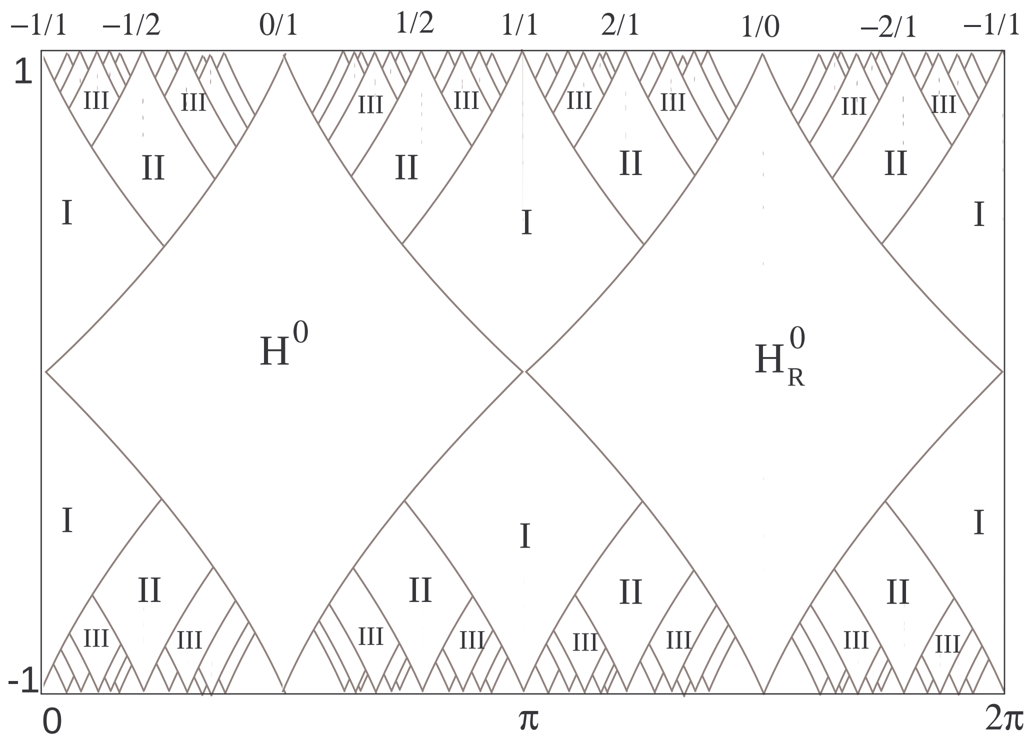


FIGURE 8. The Poincaré model of the de Sitter world: domains of first, second and third generations are indicated by I,II,III

The action of operators A, B and their inverse on the domain H^0 is shown in Figure 9, where ϕ varies from $-\pi/2$ to $3\pi/2$, so that domain H^0 is at the center of the figure.

We have subdivided H^0 into sub-domains, denoted by N,S,E,W. The operators A, B and their inverses map H^0 partially to itself, and partially outside H^0.

Remark. The corresponding actions on H^0_R are obtained taking into account the following relations among the generators, coming from (5):

$$AR = R\bar{B}; \quad BR = R\bar{A}; \quad \bar{A}R = RB; \quad \bar{B}R = RA. \tag{17}$$

Lemma 4.1. *The parts of images of H^0 (of H^0_R) by A, B and their inverse which are not in H^0 (in H^0_R) are disjoint and cover the first generation domain $\mathcal{H}^I$.*

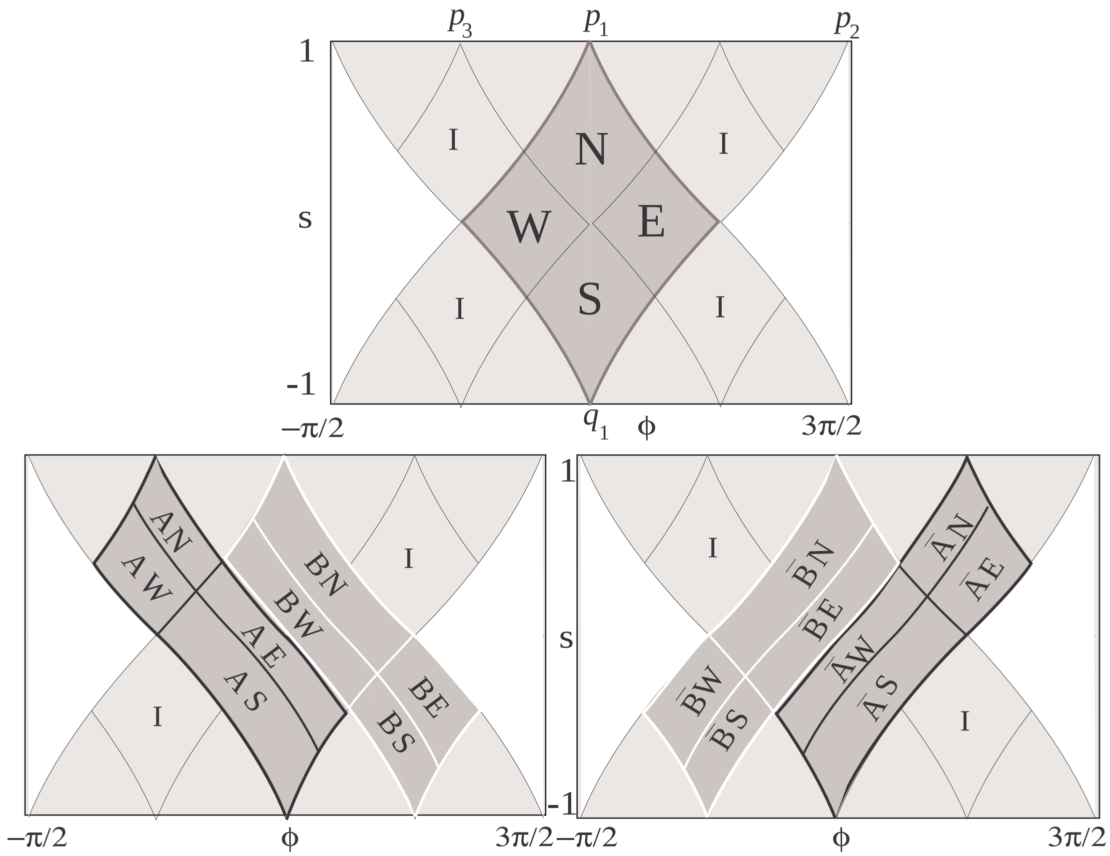


FIGURE 9. Images by A, B, $\bar{A}$ and $\bar{B}$ of H^0. Letter I indicates the four connected components of the domain of first generation $\mathcal{H}^I$

Proof. It suffices to calculate the action of the generators on the vertices of H^0, and on the intersections of the frontier lines of it with the frontiers of the rhombi of first generation. Note that the action of A and B on the points at infinite (on the circles c_1 and c_2) is given by equations (14), remembering that any point q_i on the circle c_2, symmetrical of the point p_i on c_1, as opposite vertex of the same rhombus, is the opposite ($q_i = -p'_i$) of point p'_i on the circle c_1 at distance π from p_i. So, for instance, the image under A of the extreme north p_1 of H_0 is p_3, according to (14), whereas the image under A of the extreme south, q_1, is q_1, since $q_1 = -p_2$, and $Ap_2 = p_2$ (see Figure 9). □

Definition. We denote the four connected components of $\mathcal{H}^I$, where $|S| < |K|$ and $|S| > |D|$, by (see Figure 10,left):

$$(18)\qquad \begin{aligned} H_A &= \{(K, D, S) \in \mathcal{H}^I : S > 0, K > 0\}; \\ H_{\bar{A}} &= \{(K, D, S) \in \mathcal{H}^I : S > 0, K < 0\}; \\ H_B &= \{(K, D, S) \in \mathcal{H}^I : S < 0, K < 0\}; \\ H_{\bar{B}} &= \{(K, D, S) \in \mathcal{H}^I : S < 0, K > 0\}. \end{aligned}$$

Remark. Note that the subscript index of a connected component of the first generation domain $\mathcal{H}^I$ indicates the operator mapping one part of H^0 into this connected component. Moreover

$$H_{\bar{A}} = RH_A; \quad H_{\bar{B}} = RH_B.$$

Figure 10, left, shows the four connected components of $\mathcal{H}^I$ reached from H^0 and from H^0_R by means of the operators A (black arrow) and B (white arrow). The reversed arrows must be read as the inverse operators.

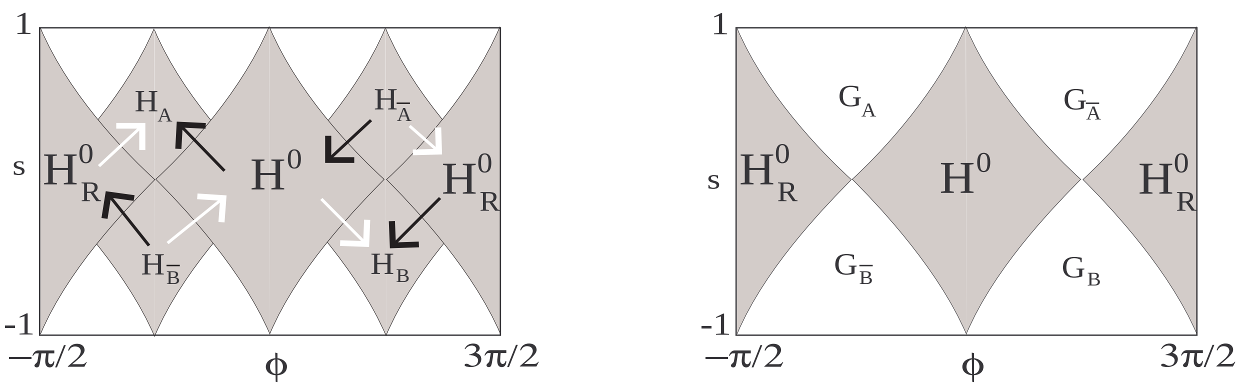


FIGURE 10.

Definition. The domain $G^0 \equiv H \setminus \overline{(H^0 \cup H^0_R)}$ consists of four disjoint connected components (see Figure 10, right), denoted by:

$$\begin{aligned} G_A &= \{(K, D, S) : S > |D|, K > 0\}; \\ G_{\bar{A}} &= \{(K, D, S) : S > |D|, K < 0\}; \\ G_B &= \{(K, D, S) : S < -|D|, K < 0\}; \\ G_{\bar{B}} &= \{(K, D, S) : S < -|D|, K > 0\}. \end{aligned}$$

Remark. $G_{\bar{A}} = RG_A$, $G_{\bar{B}} = RG_B$.

Theorem 4.2. *There is an one-to-one correspondence between the operators of $\mathcal{T}^+$ and the connected components of the domains of all generations of order $r > 0$ inside G_A (inside G_B). There is an one-to-one correspondence between the operators of $\mathcal{T}^-$ and the connected components of the domains of all generations of order $r > 0$ inside $G_{\bar{A}}$ (inside $G_{\bar{B}}$).*

In fact, for all $T \in \mathcal{T}^+$ that are the product of n generators (of type A and B), the domains TH_A, TH_B, $\bar{T}H_{\bar{A}}$, $\bar{T}H_{\bar{B}}$ cover all the connected components of the domain of the n-th generation, $\mathcal{H}^{\mathbf{n+1}}$, belonging respectively to G_A, G_B, $G_{\bar{A}}$, $G_{\bar{B}}$.

The proof of this theorem is a computation as well. The correspondence between the operators of $\mathcal{T}^+$ and the domains in G is shown in Figure 11. □

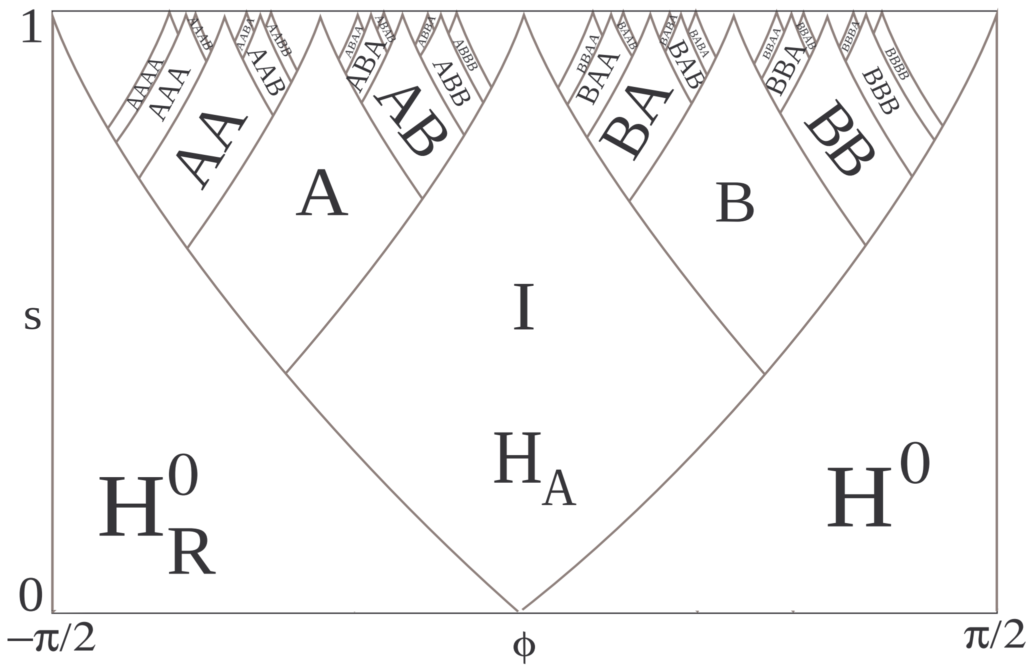


FIGURE 11.

Remark. Theorem 4.2 implies that *domains H_A and H_B behave as fundamental domains for the action of* $\mathcal{T}^+$ in G_A and G_B respectively, whereas *domains $H_{\bar{A}}$ and $H_{\bar{B}}$ behave as fundamental domains for the action of $\mathcal{T}^-$ in $G_{\bar{A}}$ and $G_{\bar{B}}$ respectively.*

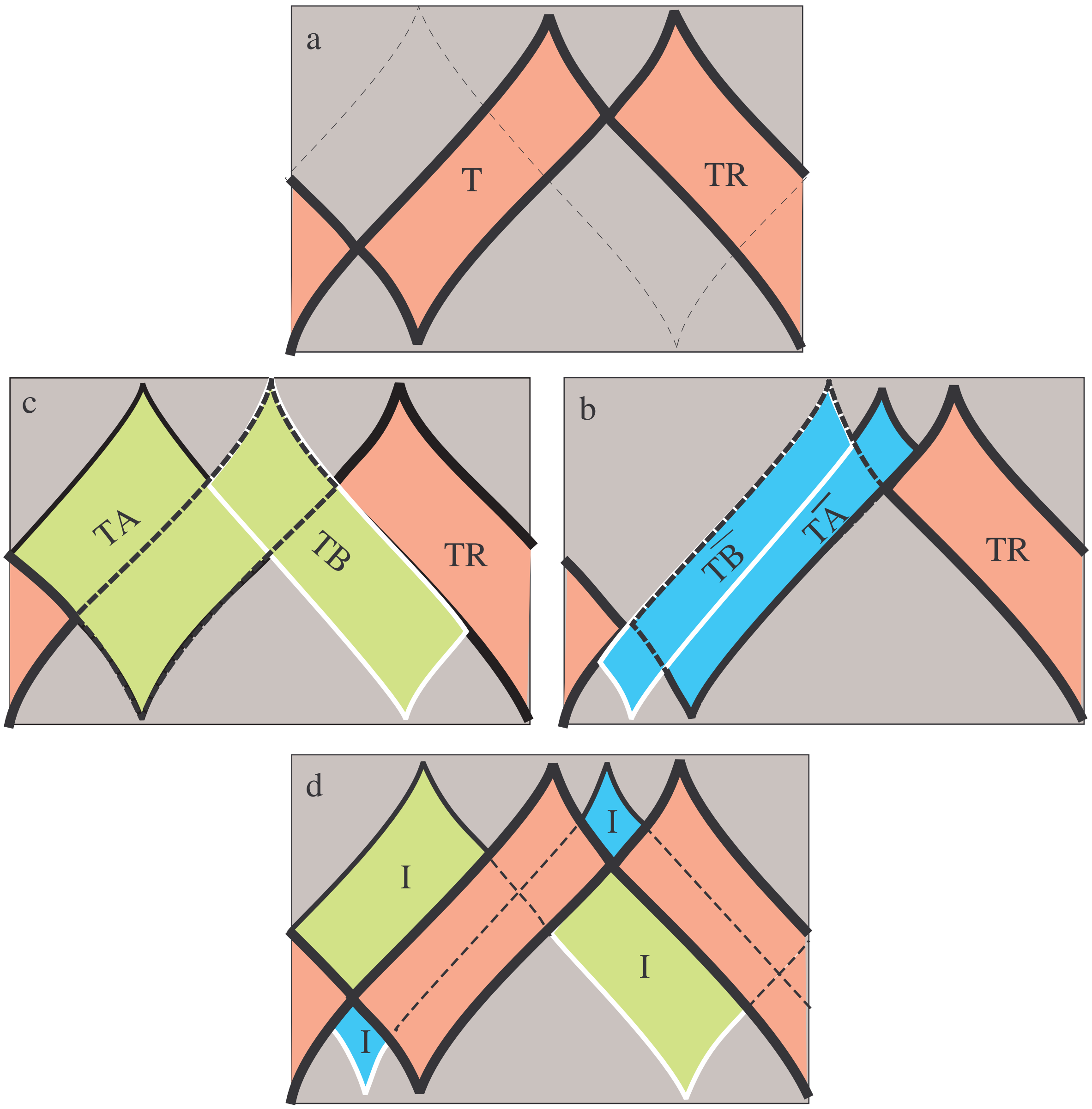
4.2. **Coordinates changes.**

Here we see how the Poincaré model of the de Sitter world changes under a change of coordinates in the space of the coefficients corresponding to a $\mathrm{SL}(2,\mathbb{Z})$ changes of coordinates in the plane where the forms are defined.

As we have said in the introduction, the situation in the de Sitter world is different from that of the Poincaré model in the Lobachevsky plane, where a change of coordinates by an operator $L \in \mathrm{SL}(2,\mathbb{Z})$ in the plane corresponds to replace the fundamental domain by its image by T_L, which is another domain of of the tiling.

The procedure in the de Sitter world is complicated by the fact that the images under $\mathcal{T}$ of the domains overlap each other, and, as we have seen in the preceding section, the key of the model is to choose, for all non-fundamental domains, the parts of them which do not overlap under the action of the semigroups $\mathcal{T}^+$ or $\mathcal{T}^-$.

Let us consider an operator $T \in \mathcal{T}$, operating our change of coordinates. The image by T of H^0 and H^0_R are the fundamental domains of the new tiling. In the following figure $T = \bar{A}$.



In part (a), the domains marked by T and TR are the image by T of H^0 and $H^0_R = RH^0$. They represents the new *rhombi of zero generation*. (The dotted lines show the boundaries of H^0 and H^0_R).

Part (b) of the figure shows the images by $T\bar{A}$ and by $T\bar{B}$ of the fundamental domain H^0 (marked by $T\bar{A}$ and $T\bar{B}$ respectively) and part (c) of the figure shows the images by TA and by TB of the fundamental domain H^0 (marked by TA and TB respectively). The union of these images form the *rhombi of the first generation*.

Part (d). To obtain the *domain of first generation* we have to exclude from these rhombi the parts of them which overlap with the rhombi of zero generation (which become the

fundamental domains H^0 and H_R^0). In this way we obtain 4 disjoint components (denoted by I), which are the image by T of the domains H_A, $H_{\bar{A}}$, H_B and $H_{\bar{B}}$ of the standard model introduced in the previous section. The procedure to build the tiling continues analogously to that already explained.

4.3. **Hyperbolic orbits.**

Figure 12 illustrates the reason of the difference between the action of $\mathcal{T}$ on the two-sheeted hyperboloid $\{E, \bar{E}\}$ and on the one-sheeted hyperboloid H, as we will explain.

Consider a generic good point $\mathbf{f}$ on a hyperboloid in the space (K, D, S).

Definition. The sequences of points $[A^j\mathbf{f}]$ and $[B^j\mathbf{f}]$, where $j \in \mathbb{Z}$, are called *neck-laces* and are denoted by $\omega_{\mathbf{f}}(A)$ and $\omega_{\mathbf{f}}(B)$, respectively. The ordering of $\mathbb{Z}$ provides any neck-lace with an orientation.

Proposition 4.3. *Every neck-lace $\omega_{\mathbf{f}}(A)$ (every neck-lace $\omega_{\mathbf{f}}(B)$) lies on the intersection of the hyperboloid with a plane of the family $S = -D + \alpha$ (respectively, $S = D + \alpha$), $\alpha \in \mathbb{Z}$.*

Proof. For every $\mathbf{f} = (K, D, S)$, the vector $(A-I)\mathbf{f}$ is orthogonal to the vector $(0, 1, 1)$ (the normal vector to the family of planes $S+D-\alpha = 0$) and the vector $(B-I)\mathbf{f}$ is orthogonal to the vector $(1, -1, 0)$ (the normal vector to the family of planes $S - D - \alpha = 0$). $\square$

Remark. The notion of neck-lace is independent of the type (elliptic or hyperbolic) of orbits. Figure 12 shows some lines where the neck-laces lie on both two-sheeted and one-sheeted hyperboloids. Black colour is used for neck-laces of type $\omega_{\mathbf{f}}(A)$ and white colour for neck-laces of type $\omega_{\mathbf{f}}(B)$. The arrow shows the orientation of the neck-lace.

Note that in the elliptic case the projections of the neck-laces lie on horocycles tangent to the circle C (the ∞) at the points $\widetilde{K} = 0$ (p_1 and p_2 in Figure 12**b**).

Definition. We call *semi-orbits* $O^+_{\mathbf{f}}$ and $O^-_{\mathbf{f}}$ the sets of good points reached by the action of the semigroups $\mathcal{T}^+$ and $\mathcal{T}^-$, respectively, excluding the identity, on a good point $\mathbf{f}$.

For every $\mathbf{f}$ in E, $O^+_{\mathbf{f}}$ never contains $\mathbf{f}$. This can be seen starting by any point in E, and trying to reach it by a path composed of pieces of neck-laces always in the same direction of the arrow (i.e., by an operator either of $\mathcal{T}^+$, or of $\mathcal{T}^-$). In H the situation is different: a series of good points obtained one form the previous one applying in sequence either operator A or operator B (i.e., forming a path composed of pieces of neck-laces in the positive direction) can lie on a cycle (see for instance the dotted-line in Figure 12**d**).

We will consider separately the case when Δ is a square number.

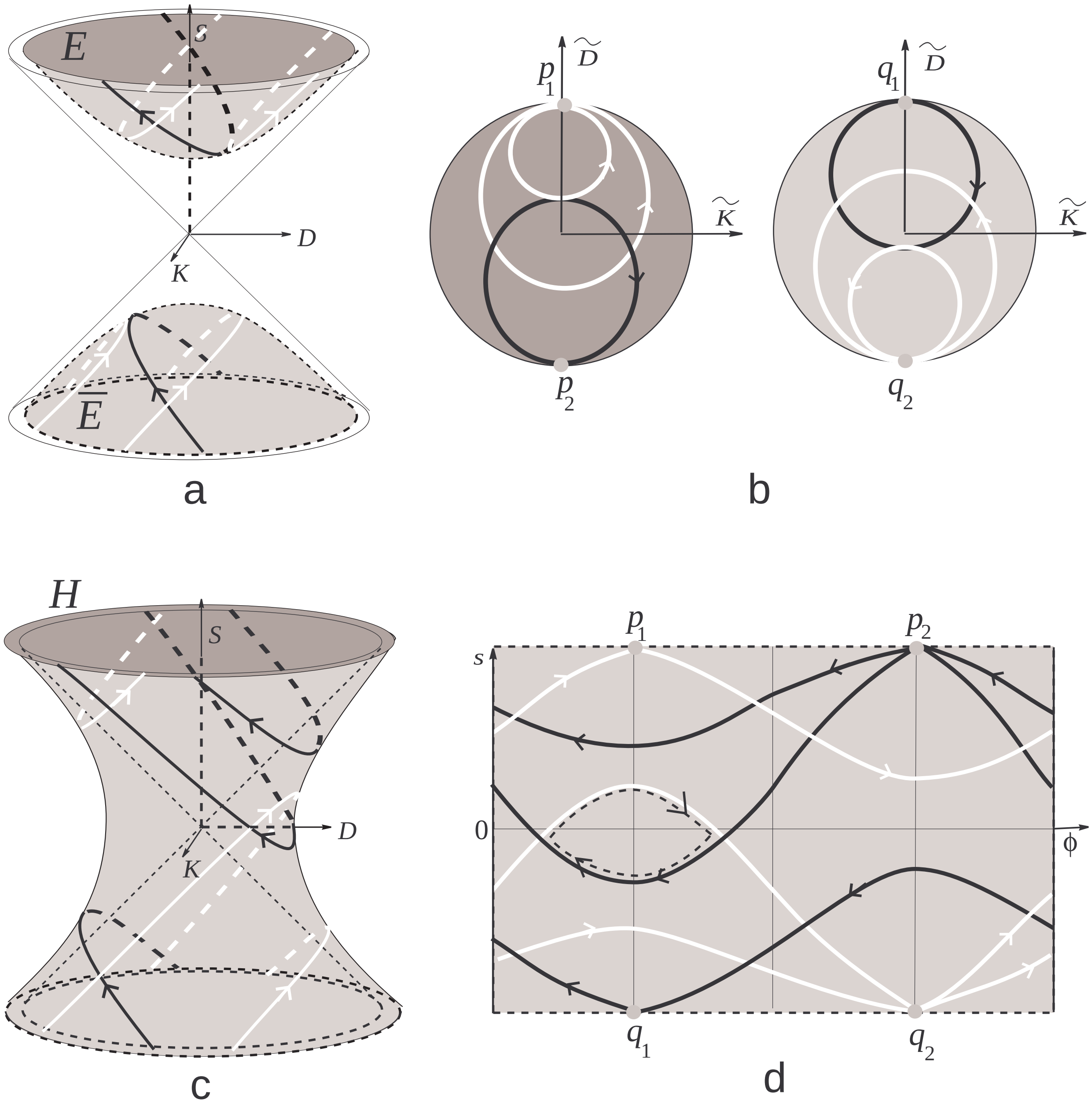


FIGURE 12. a) Neck-laces on the two-sheeted hyperboloid $\{E, \bar{E}\}$; b) their projection on the Lobachevsky disc; c) Neck-laces on the one-sheeted hyperboloid H; d) their projection on the cylinder C_H

4.4. **The case of Δ different from a square number.**

Theorem 4.4. *For every Δ different from a square number there exists some good point $\mathbf{f}$ on the hyperboloid H_Δ and an operators T of $\mathcal{T}^+$ $(\mathcal{T}^-)$, such that $T\mathbf{f} = \mathbf{f}$. Such a point belongs to H^0.*

Proof. The proof follows from some lemmas.

Lemma 4.5. *The domain H^0 (H^0_R) contains a finite number of good points.*

Proof. Since $mn = (S^2 - D^2)/4$, the domain H^0 contains all forms where $m > 0$ and $n < 0$, whereas H^0_R contains all forms where $m < 0$ and $n > 0$. From the definition of Δ we obtain

$$4mn = k^2 - \Delta.$$

Since in H^0 the product mn is negative, the above equality is fulfilled by a finite number of values of k, less than $\sqrt{\Delta}$. For each one of these values a finite number of product $|4mn|$ equals $\Delta - k^2$. $\square$

Note that the condition $\sqrt{\Delta} \notin \mathbb{Z}$ means that m and n cannot vanish, hence we have $|D| \neq |S|$. Thus G^0 ($G^0 \equiv H \setminus (\overline{H^0 \cup H^0_R})$) contains all hyperbolic forms where m and n have the same sign.

Definition. A *cycle of length t* ($t > 1$) is a sequence of points $[\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_t]$ such that $\mathbf{f}_i = T_{i-1}\mathbf{f}_{i-1}$ ($i = 2, \dots, t$) and $\mathbf{f}_1 = T_t\mathbf{f}_t$, where each one of operators $T_1, T_2, \dots, T_t$ is either A or B. A cycle of length t is indicated by $\gamma_{\mathbf{f}}(T_1, \dots, T_t)$ where $\mathbf{f} = \mathbf{f}_1$. An equivalent notation of the cycle $\gamma_{\mathbf{f}}(T_1, \dots, T_t)$ is, evidently, $\gamma_{\mathbf{g}}(T_j, \dots, T_t, T_1, \dots T_{j-1})$, where $\mathbf{g} = \mathbf{f}_j$.

Lemma 4.6. *Every good point $\mathbf{f}$ in H^0 satisfies: $A\mathbf{f} \in H^0$ iff $B\mathbf{f} \in H_B$; $B\mathbf{f} \in H^0$ iff $A\mathbf{f} \in H_A$, and, moreover, $\bar{A}\mathbf{f} \in H^0$ iff $\bar{B}\mathbf{f} \in H_{\bar{B}}$; $\bar{B}\mathbf{f} \in H^0$ iff $\bar{A}\mathbf{f} \in H_{\bar{A}}$. Analogous statements hold substituting H^0 with H^0_R.*

Proof. By the action of A, the point $\mathbf{f}$, in coordinates (m, n, k), is sent to $A\mathbf{f} = (m, m + n + k, 2m + k)$, and, by B, to $B\mathbf{f} = (m + n + k, n, 2n + k)$. Since $\mathbf{f}$ belongs to H^0, $m > 0$ and $n < 0$. Now, if $m + n + k > 0$ then $A\mathbf{f}$ belongs to G^0, and $B\mathbf{f}$ belongs to H^0, whereas if $m + n + k < 0$ then $A\mathbf{f}$ belongs to H^0, and $B\mathbf{f}$ to G^0. Similar inequalities hold for the inverse generators. By Lemma 4.1, the image by A, B, $\bar{A}$ and $\bar{B}$ of a point in H^0, if it is in G^0, belongs respectively to H_A, H_B, $H_{\bar{A}}$, and $H_{\bar{B}}$. Because of relations (17), analogous arguments hold for H^0_R. $\square$

Lemma 4.7. *For every good point $\mathbf{f}$ of H^0 (H^0_R) there is an integer $t > 1$ such that $\mathbf{f}$ belongs to a unique cycle $\gamma_{\mathbf{f}}(T_1, \dots, T_t)$.*

Proof. Let $\mathbf{f}_1 = \mathbf{f}$. By Lemma 4.6, either $A\mathbf{f}_1$ or $B\mathbf{f}_1$ belongs to H^0. Let $\mathbf{f}_2 = T_1\mathbf{f}_1$ be in H^0, being $T_1 = A$ or $T_1 = B$. Now, let $\mathbf{f}_3$ be the point, among $A\mathbf{f}_2$ and $B\mathbf{f}_2$, which belongs to H^0, etc. We find in this way a sequence of points $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \dots$ in H^0. Since, by Lemma 4.5, H^0 contains only a finite number of good points, the sequence of point

$[\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_t]$ must be periodic, i.e., we find eventually a cycle $\gamma_{\mathbf{f}}(T_1, \dots, T_t)$. For H^0_R the proof is analogous. $\square$

Lemma 4.8. *Different cycles are disjoint.*

Proof. By Lemma 4.6, any point of a cycle determines the others, hence if two cycles have a common point, then they coincide. $\square$

Lemma 4.9. *H^0 (H^0_R) contains at least one good point if $\sqrt{\Delta} \notin \mathbb{Z}$.*

Proof. The discriminant, Δ, either is divisible by 4 or $\Delta = 4d + 1$. If $\Delta = 4d$, then the point $(m = d, n = -1, k = 0)$ belongs to H^0 and the point $(m = -1, n = d, k = 0)$ belongs to H^0_R. If $\Delta = 4d + 1$, then the point $(m = d, n = -1, k = 1)$ belongs to H^0 and the point $(m = -1, n = d, k = 0)$ belongs to H^0_R. $\square$

We have thus completed the proof of the Theorem[4] 4.4. Indeed, let $\mathbf{f}$ be a good point of H^0, defined by Lemma 4.9. By Lemma 4.7, it belongs to a cycle $\gamma_{\mathbf{f}}(T_1, \dots, T_t)$. Hence $T = T_t T_{t-1} \cdots T_2 T_1 \in \mathcal{T}^+$ satisfies $T\mathbf{f} = \mathbf{f}$. $\square$

Theorem 4.10. *There is an one-to-one correspondence between the orbits under $\mathcal{T}$ of the good points of H_Δ (when Δ is different from a square number) and the cycles in H^0.*

Proof. The point $R\mathbf{f}$, obtained from $\mathbf{f}$ by a shift by π on the cylinder C_H, belongs to the orbit of $\mathbf{f}$. Hence any orbit on C_H is invariant under a shift by π. All the following statements on H^0 hold analogously for H^0_R.

By Lemma 4.8, different cycles are disjoint. Every cycle belongs to some orbit, by definition of orbit. We have to prove: a) that every orbit contains a cycle; b) that this cycle is unique.

a) Let $\mathbf{f}$ be a point, non belonging to a cycle, so $\mathbf{f} \in G^0$. Let us suppose that $\mathbf{f} \in G_A$ (in the other cases the proof is analogous). By Theorem 4.2, there exists a unique operator T of $\mathcal{T}^+$ such that $\mathbf{g} = T^{-1}\mathbf{f}$ belongs to H_A. Hence $\mathbf{h} = \bar{A}\mathbf{g} = \bar{A}T^{-1}\mathbf{f}$ is inside H^0. But if a point belongs to H^0, then it belongs to a cycle by Lemma 4.7, and hence the orbit of $\mathbf{f}$ contains a cycle.

b) We have to prove that the point $\mathbf{h}$ in H^0, obtained from $\mathbf{f}$ by the above procedure, is unique, i.e., that we cannot reach another point of H^0 non belonging to the cycle $\gamma_{\mathbf{h}}$ by an operator of $\mathcal{T}$. By Lemma 1.1, every operator of $\mathcal{T}$ can be written as USV, where S belongs to $\mathcal{T}^+$ or to $\mathcal{T}^-$, and V and U are equal to the identity or to the operator R.

[4] An alternative proof of this theorem should consist in proving that the set of eigenvectors corresponding to the eigenvalue $\lambda = 1$ of the operators of $\mathcal{T}^+$ contains an integer good vector (v_1, v_2, v_3) such that $v_1^2 + v_2^2 - v_3^2 = 4d + e$, for every $d \in \mathbb{N}$ and $e \in \{0, 1\}$ whenever $4d + e$ is different from a square number.

Hence we try to reach H^0 from $\mathbf{f}$ by means of operators of such types. We have to start by R, reaching a point $\mathbf{p}$ of $G_{\bar{A}}$. As before, there is only one operator of $\mathcal{T}^-$ such that p is the image by it of a point in $H_{\bar{A}}$, indeed:

$$\begin{aligned} &\mathbf{p} = R\mathbf{f} \quad \mathbf{p} \in G_{\bar{A}}, \quad \text{hence} \\ &\mathbf{p} = R\,T\mathbf{g} = \hat{T}R\mathbf{g} \quad , \end{aligned}$$

where $\hat{T}$ is the operator obtained by T replacing each A by $\bar{B}$ and each B by $\bar{A}$, and vice versa. The operator $\hat{T}$ belongs to $\mathcal{T}^-$, and, being $\mathbf{g} \in H_A$, point $\mathbf{j} = R\mathbf{g}$ is in $H_{\bar{A}}$. Now we can reach a point either in H^0 (by A), or in H^0_R (by B). Since $g = A\mathbf{h}$, we obtain

$$B\mathbf{j} = RR\;B\mathbf{j} = R\;\bar{A}R\;R\mathbf{g} = R\;\bar{A}A\mathbf{h} = R\mathbf{h}.$$

Therefore in this case the point in H^0_R, reached from $\mathbf{p}$ in $G_{\bar{A}}$, is exactly $R\mathbf{h}$. On the other hand, $A\mathbf{j} \in H^0$ is equal to:

$$A\mathbf{j} = RR\;A\mathbf{j} = R\;\bar{B}R\;R\mathbf{g} = R\;\bar{B}\;A\mathbf{h} = AR\;A\mathbf{h} = B\mathbf{h}.$$

Since $A\mathbf{h}$ is in H_A, $B\mathbf{h}$ is in H^0 (by Lemma 4.6) and belongs to the orbit of $\mathbf{h}$. The proof is completed. $\square$

Figure 14 shows examples of orbits projected to the cylinder C_H.

Theorems 4.4 and 4.10 implies the following two Corollaries.

Corollary 4.11. *The set of goods points in H^0 (H^0_R) is partitioned into cycles.*

Corollary 4.12. *Let operator T of $\mathcal{T}^+$ ($\mathcal{T}^-$) satisfy $T\mathbf{f} = \mathbf{f}$, for some good point $\mathbf{f} \in H^0$, and T be composed by t generators of type A or B. Then in H_0 the orbit of $\mathbf{f}$ has t points $\mathbf{f}_i$, including $\mathbf{f}$, satisfying $\tilde{T}_i\mathbf{f}_i = \mathbf{f}_i$, $\tilde{T}_i$ ($i = 1, \ldots, t$) being obtained from T by a cyclic permutation of the sequence of the t generators defining it. Such points belong to H^0, and no other points of the same orbit belong to H^0.*

Theorem 4.13. *The operator T defining the cycle in H^0 is the product of t operators, t_A of type A and t_B of type B ($t_A + t_B = t$) iff:*

– in H_A and in $H_{\bar{A}}$, as well as in every domain in G_A and $G_{\bar{A}}$, there are t_B points;

– in H_B and in $H_{\bar{B}}$, as well as in every domain in G_B and $G_{\bar{B}}$, there are t_A points.

Proof. By Theorem 4.4, the set of points of every cycle in H^0 is subdivided into two disjoint subsets: the set of points whose image under A belongs to H_A and the set of points whose image under B belongs to H_B. By Lemma 4.6, the image by A of a point is inside H_A if and only if its image under B is inside H^0. Moreover, there are t_B of such points, if and only if T contains t_B generators of type B (see Figure 13 and Figure 10). Similarly, the image by B of a point is inside H_B if and only if the image under A is inside

H^0; and there are t_A of such points, if and only if T contains t_A generators A. Similarly, in virtue of the same Lemma 4.6, we obtain an analogous statement, considering the inverse of the operator T: the image by $\bar{A}$ of a point is inside $H_{\bar{A}}$ if and only if the image under $\bar{B}$ is inside H^0; and there are t_B of such points, if and only if $\bar{T}$ contains t_B generators of type $\bar{B}$, etc. (see Figure 13). $\square$

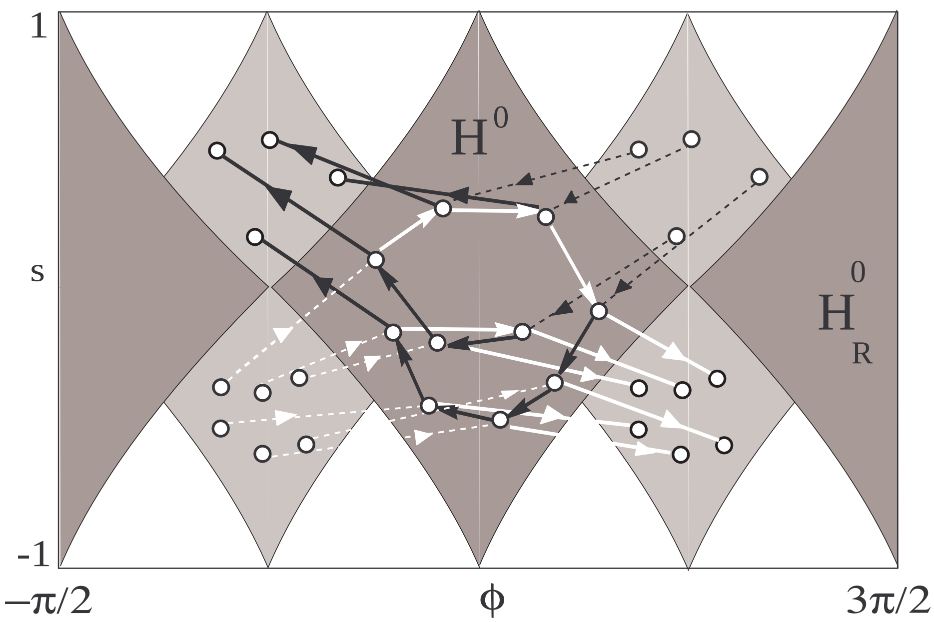


FIGURE 13. The cycle in H^0 of an asymmetric orbit, for $\Delta = 624$, containing 10 points, among which four are mapped by A to H_A, six by B to H_B, four by $\bar{A}$ to $H_{\bar{A}}$ and six by $\bar{B}$ to $H_{\bar{B}}$

4.5. **The case of Δ equal to a square number.**

This case is essentially different from the preceding one because the coefficients m and n of the form can vanish. This means that for such forms either $S = D$ or $S = -D$, and therefore the corresponding good points lie on the boundary of H^0 and of H^0_R.

Theorem 4.14. *On the hyperboloid H_Δ, $\Delta = \rho^2$, $\rho \in \mathbb{Z}$, there are exactly ρ orbits, corresponding to the points $(K = \rho,\ D = r,\ S = r)$, $r = 0 \ldots \rho - 1$.*

Proof. The proof follows from the following lemmas.

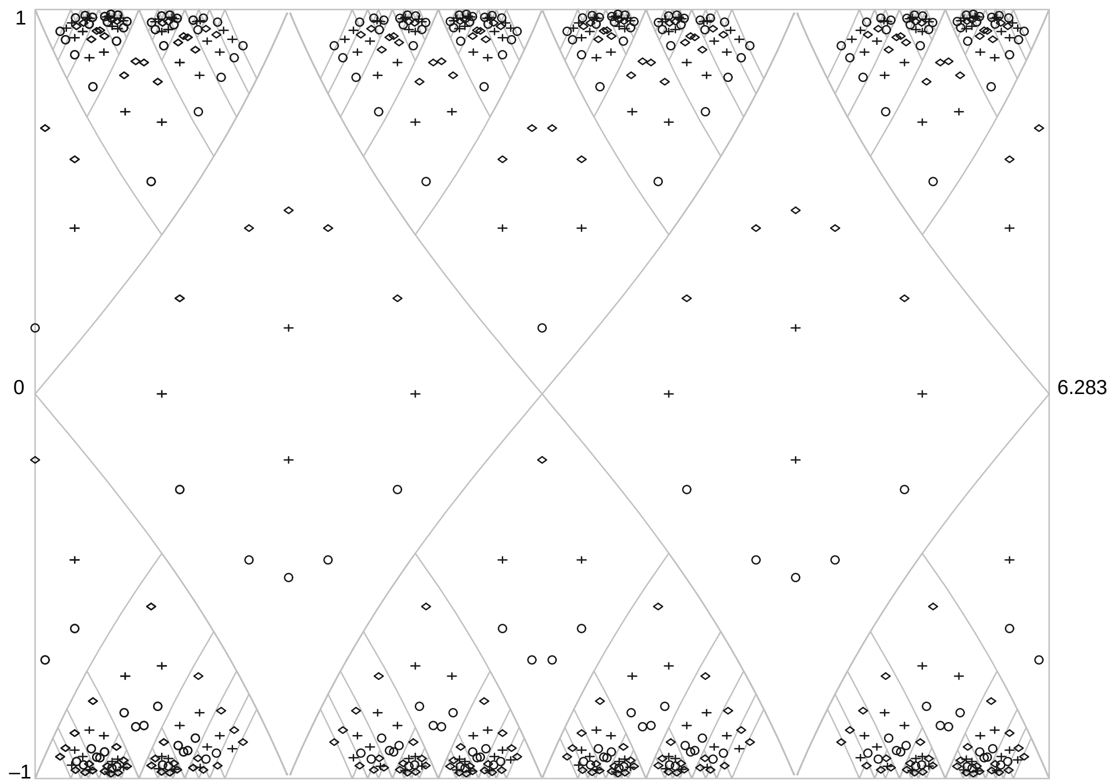


FIGURE 14. The initial parts of the 3 distinct orbits with $\Delta = 32$ projected to the cylinder. The initial points of orbits in coordinates (K, D, S) are: (-6,0,2), circles, k-symmetric orbit; (6,0,2), diamonds, k-symmetric orbit; (-4,4,0), crosses, supersymmetric orbit.

Definition. We use the following notations for the open segments, belonging to the common frontier of H^0 and H_A, $H_{\bar{A}}$, H_B and $H_{\bar{B}}$:

$$\begin{aligned} F_A = \ & \{(K = \rho, D = r, S = r), 0 < r < \rho\} \ ; \\ F_{\bar{A}} = \ & \{(K = -\rho, D = r, S = r), 0 < r < \rho\} \ ; \\ F_{\bar{B}} = \ & \{(K = \rho, D = r, S = -r), 0 < r < \rho\} \ ; \\ F_B = \ & \{(K = -\rho, D = r, S = -r), 0 < r < \rho\}. \end{aligned}$$

Lemma 4.15. *There is an one-to-one correspondence between the images of the set F_A by the operators of $\mathcal{T}^+$ which are product of n generators of type A and B and the lower-right sides of the frontiers of all domains of the $(n+1)$-th generation inside G. There is an one-to-one correspondence between the images of set F_B by the operators of $\mathcal{T}^+$ (product of n generators) and the upper-left sides of the frontiers of all domains of the $(n+1)$-th generation inside G_B. (Similar statements hold for $\mathcal{T}^-$, $F_{\bar{A}}$ and $F_{\bar{B}}$).*

Proof. F_A is the lower-right side of the frontier of H_A. The action of $\mathcal{T}^+$ on F_A is deduced from that on H_A (see Theorem 4.2). Note that F_B is the upper-left side of the frontier of H_B, etc. □

Lemma 4.16. *Every good point* $\mathbf{f}$ *in* H^0 *satisfies:*

$A\mathbf{f} \in H^0$ *iff* $B\mathbf{f} \in H_B$; $B\mathbf{f} \in H^0$ *iff* $A\mathbf{f} \in F_A$;

$\bar{A}\mathbf{f} \in H^0$ *iff* $\bar{B}\mathbf{f} \in H_{\bar{B}}$; $\bar{B}\mathbf{f} \in H_0$ *if* $\bar{A}\mathbf{f} \in H_{\bar{A}}$.

Moreover, $A\mathbf{f} \in F_A$ *iff* $B\mathbf{f} \in F_B$ *and* $\bar{A}\mathbf{f} \in F_{\bar{A}}$ *iff* $\bar{B}\mathbf{f} \in F_{\bar{B}}$.

Proof. This lemma is the version of Lemma 4.6 when Δ is equal to a square number. Indeed, if $\mathbf{f} = (m, n, k)$ and $\mathbf{g} = A\mathbf{f} \in F_A$, then $(m + n + k = 0)$. This implies that $\mathbf{g}' = B\mathbf{f} = (m + n + k, n, k + 2n)$ belongs to F_B. The cases of the inverse generators are similar. □

Lemma 4.17. *For every* $\Delta = \rho^2$*, the orbit of the point* $(K = \rho, D = 0, S = 0)$ *is supersymmetric: it contains, with point* $(-\rho, 0, 0)$*, all the lower points of the frontier of all domains in* G *and* $G_{\bar{A}}$ *and all upper points of the frontiers of all domains in* G_B *and in* $G_{\bar{B}}$*. These points are the only points of the orbit.*

Proof. This lemma is a consequence of Lemma 4.1 and 4.2, being points $(K = \pm\rho, D = 0, S = 0)$ the lower points of domains H_A and $H_{\bar{A}}$ and the upper points of domains H_B and $H_{\bar{B}}$. □

To conclude the proof of Theorem 4.14, remember that, by Lemma 4.1, all good points inside the interior of domains H_Z $(Z = A, B, \bar{A}, \bar{B})$ are image by Z of points at the interior of H^0. So, we are now interested in the images under Z^{-1} of the points of F_Z which are inside H^0. For instance, we start from a point $\mathbf{h} \in F_{\bar{A}}$ (see Figure 15) and we go to $\mathbf{f} \in H^0$ applying A. Afterwards, we apply in sequel either A or B in order to remain inside H^0 till we reach point $\mathbf{g}$, such that both $A\mathbf{g}$ and $B\mathbf{g}$ belong, by Lemma 4.16, to the frontier of H^0, namely to F_A and F_B. Lemma 4.16 says also that $\bar{B}\mathbf{f}$ belongs to $F_{\bar{B}}$, since $\bar{A}\mathbf{f}$ belongs to $F_{\bar{A}}$. By a similar procedure we associate to every point $\mathbf{h}$ of any one of sets F_Z, a chain of points inside H^0 and three other points of the orbit of $\mathbf{f}$, one in everyone of the other sets F_Y, $Y \neq Z$ (see Figure 15). In his way we associate to any point inside H^0 a unique ordered chain of points inside H^0, whose initial point is mapped, by $\bar{A}$ and $\bar{B}$ to two points of $F_{\bar{A}}$ and $F_{\bar{B}}$, respectively, and whose final point is mapped, by A and B to two points of F_A and F_B, respectively (see figure 15). Since also in this case different chains cannot have common elements, we have $\rho - 1$ distinct orbits, corresponding to all integer points in $F_{\bar{A}}$ $((\rho, r, r)$ for $r = 1 \ldots \rho - 1)$, plus the orbit of point $(\rho, 0, 0)$, given by Lemma 4.17. □

The above theorem has this immediate corollary:

Corollary 4.18. *The good points in H^0 are partitioned into disjoint chains whose points are obtained one from the preceding point by A or by B. The final point is sent by A to F_A, by B to F_B, whereas the initial point is sent by $\bar{A}$ to $F_{\bar{A}}$ and by $\bar{B}$ to $F_{\bar{B}}$. Every chain corresponds to one orbit.*

Figure 16 shows the case $\Delta = 25$.

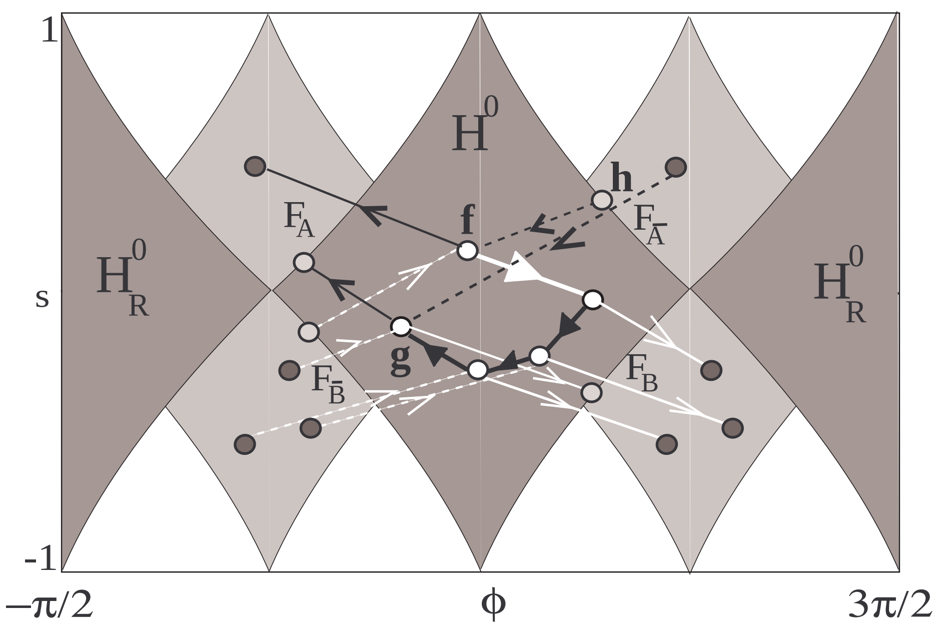


FIGURE 15. Chain of points inside H^0 and its terminating points on the frontier, for $\Delta = 81$. Black arrow: operator A, white arrow: operator B. The orbit is asymmetric

Theorem 4.19. *Let t be the number of points of a chain inside H^0, with initial point $\mathbf{f}$ and final point $\mathbf{g}$. Let T be the operator of $\mathcal{T}^+$ satisfying $T\mathbf{f} = \mathbf{g}$. Then T is the product of $t-1$ generators. Among them, t_A are of type A and t_B of type B, iff the orbit of $\mathbf{f}$ contains exactly t_B points inside every domain in G_A and $G_{\bar{A}}$, and exactly t_A points inside every domain in G_B and $G_{\bar{B}}$, so that $t_A + t_B = t - 1$.*

Proof. The points $\mathbf{f} = \mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_t = \mathbf{g}$ of the chain are mapped one to the successive one by operator A or B. By Lemma 4.16, to every image by A inside the chain there is an image by B in H_B and for every image by B inside the chain there is an image by A in H_A. Considering the reversal chain, where point $\mathbf{f}$ is reached from $\mathbf{g}$ by t_A operators $\bar{A}$ and t_B operators $\bar{B}$, we obtain that there are t_A points inside $H_{\bar{B}}$ and t_B points in $H_{\bar{A}}$. To complete the proof we use Theorem 4.2. $\square$

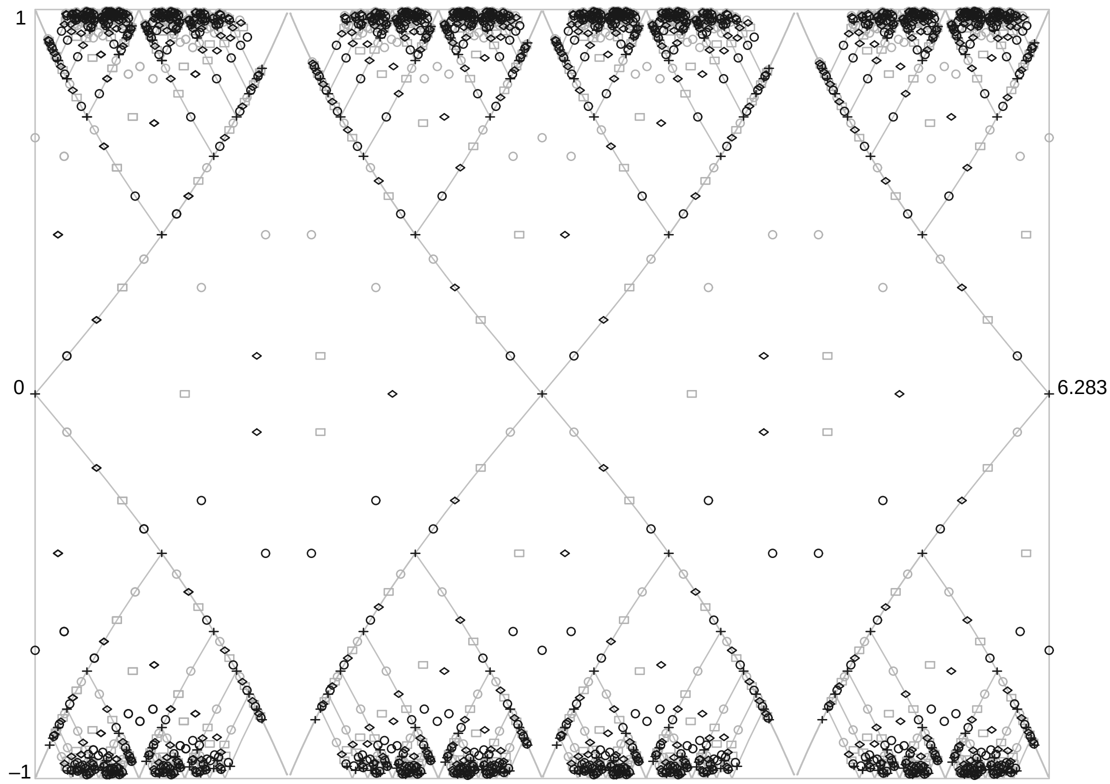


FIGURE 16. Initial part of the 5 distinct orbits for $\Delta = 25$ projected on the cylinder. The initial points of orbits are, in coordinates (K, D, S): (5,0,0), crosses, supersymmetric orbit; (5,1,1), black circles, k-symmetric orbit; (5,2,2), black diamonds, $(m+n)$-symmetric orbit; (5,3,3), gray boxes, $(m+n)$-symmetric orbit; (5,4,4), gray circles, k-symmetric orbit

Example. In Figure 16, the orbit of (5,1,1) (black circles) has a chain in H_0 composed of 4 points. The 3 operators between them are all of type A, since there are no black circles in H_A and $H_{\bar{A}}$, and thus $t_A = 3$ and $t_B = 0$. Vice versa for the orbit of (5,4,4) (gray circles) $t_A = 0$ and $t_B = 3$. The orbits of (5,2,2) and (5,3,3), contain in H^0 a chain of 3 points (diamonds and boxes, respectively), and for both orbits $t_A = t_B = 1$, since there is a diamond and a box inside every domain, in regions $G, G_{\bar{A}}, G_B$ and $G_{\bar{B}}$.

5. Final remarks

5.1. **Remark on the behaviour of the non rational points under $\mathcal{T}$.** Note that on the hyperboloid $K^2 + D^2 - S^2 = \Delta$ the orbit of any point with fractional coordinates (K, D, S), having the common denominator μ, is obtained (by a scale reduction) from the orbit of the good point with integer coordinates $(2\mu K, 2\mu D, 2\mu S)$ on the hyperboloid

with discriminant $4\mu^2\Delta$, and hence has a finite number of points in H^0, in H^0_R as well as in every domain of any generation.

Differently from the elliptic case, where the orbit under $\mathcal{T}$ of an irrational point is described exactly as that of an integer point (close points in the Lobachevsky disc have close orbits under $\mathrm{PSL}(2,\mathbb{Z})$), in the hyperbolic case the situation is completely different. Indeed, Theorems 4.2 and 4.4 imply that two close points in H have close semi-orbits only if these points belong to the complement of H^0 and H^0_R. But the orbits – and even the semi-orbits – of two close points of H^0 or H^0_R are not close (this follows from the fact that an analogous statement as Lemma 4.6 holds for all points – not only for the good points – in H^0 and in H^0_R).

Moreover, for the irrational points on the one-sheeted hyperboloid we obtain, from Theorems 4.2 and 4.4, the following

Corollary 5.1. *The orbit of any point having at least one irrational coordinate contains an infinite number of points in H^0 and in H^0_R (and hence in each connected component of the domains of all generations of the hyperboloid).*

5.2. **Sun eclipse model of the de Sitter world.**

In this section we see the Poincaré model of the de Sitter world under an alternative projection.

Consider the domains of different generations directly on the hyperboloid H. The lines bounding such domains belong to the straight lines generatrices of the hyperboloid. More precisely, in the plane $S = 1$ the segment joining the point p_i and its opposite point on the circle c_1, upper vertices of a pair of rhombi of the n-th generation, defines a direction on the plane $S = 1$. On the plane $S = 0$, consider two straight lines l_1 and l_2 in such direction tangent to the circle of radius $\rho = \sqrt{\Delta}$, intersection of H with that plane. The four generatrices of the hyperboloid bounding the domains projected by $\mathcal{Q}$ to these rhombi are the intersection of the hyperboloid with two vertical planes through l_1 and l_2.

The hierarchy of points p_i is inherited by the pairs of parallel lines on the plane $S = 0$ as well as by the regions bounded by such pairs of lines and by the circle $K^2 + D^2 = \rho^2$. The regions of the n-th generation *lie behind* those of all preceding generations. The view of the domains on the hyperboloid projected to the plane $S = 0$ is shown in Figure 17. Note that by this projection we map only one half of H.

To introduce the 'sun eclipse model', let us consider a property of projection $\mathcal{P}$ (equation 8).

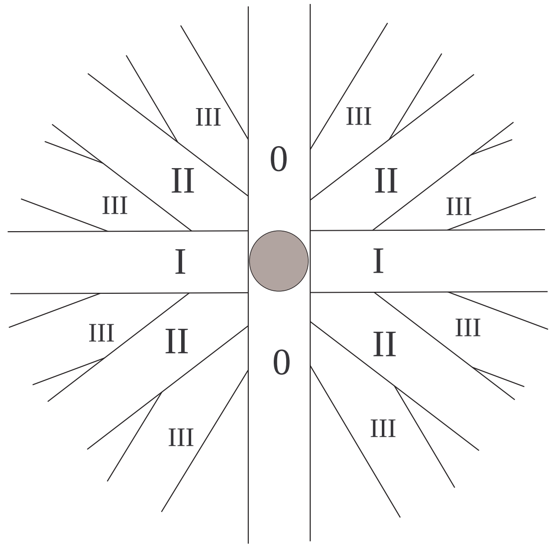


FIGURE 17.

Definitions. Let $\mathbf{f}$ be a point of the upper sheet E of the two-sheeted hyperboloid ($K^2 + D^2 = S^2 + \Delta$, $\Delta < 0$, see Figure 18). Let $\mathcal{P}'$ be the projection of E to the plane $S = \rho = \sqrt{-\Delta}$ along the vertical direction, and $\mathbf{r} = \mathcal{P}'\mathbf{f}$. Let $\mathcal{P}''$ be the projection of the plane $S = \rho$ to the sphere of radius ρ from the centre of coordinates, and $\mathbf{s} = \mathcal{P}''\mathbf{r}$. Let $\mathcal{P}'''$ be the stereographic projection of the upper half-sphere to the disc of radius ρ in the plane $S = 0$ from the point O' ($K = D = 0$, $S = -\rho$), and $\mathbf{g} = \mathcal{P}'''\mathbf{s}$.

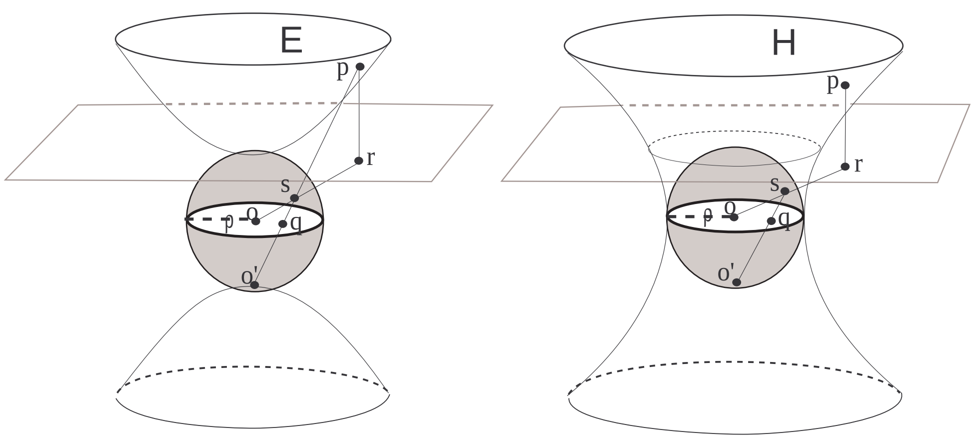


FIGURE 18. Point p on the hyperboloid is sent to q on the disc by a projection which results by the composition of three projections

Proposition 5.2. *Point* $\mathbf{q} = \mathcal{P}'''\mathcal{P}''\mathcal{P}'\mathbf{p}$ *coincides with the image of the projection* $\mathcal{P}\mathbf{p}$ *of* $\mathbf{p}$ *directly to the plane* $S = 0$ *from point* O'.

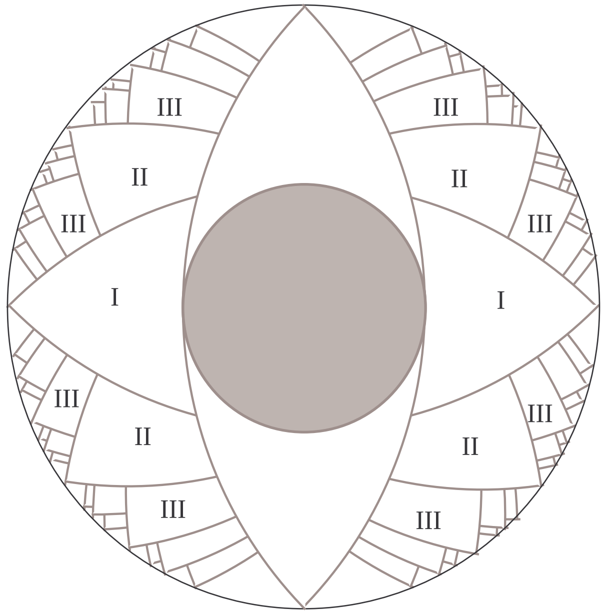


FIGURE 19. Sun-eclipse model of the de Sitter world

Proof. Calculation. □

The projection of the upper half-hyperboloid H ($K^2 + D^2 = S^2 + \Delta$, $\Delta > 0$) to the plane $S = 0$ from point O' is not convenient at all: (it is two-to-one in a ring contained in the unitary disc. However, projections $\mathcal{P}'$, $\mathcal{P}''$ and $\mathcal{P}'''$ (and their symmetric ones for the lower half-hyperboloid) are well defined (see Figure 18). We thus project the upper half-hyperboloid H by $\mathcal{P}'''\mathcal{P}''\mathcal{P}'$ to the disc of radius ρ to the plane $S = 0$. The image is contained in the ring $\frac{\rho}{2} <= \sqrt{K^2 + D^2} < \rho$. The pairs of straight lines tangent to the disc $K^2 + D^2 = \rho^2$ in the plane $S = \rho$ are projected by $\mathcal{P}''$ to half meridian circles of the sphere of radius ρ, and hence, by the stereographic projection $\mathcal{P}'''$, to arcs of circles tangent to the circle $\sqrt{k^2 + D^2} = \rho/2$. The final disc of unit radius is obtained by rescaling. On the boundary C of this disc the same points p_i considered at the boundary of the Lobachevsky disc are the extreme points of the domains of all generations (forming the sun corona). The empty (black) disc of radius 1/2 is the moon in the sun-eclipse model (see Figure 19).

Remark. The complementary forms (symmetric with respect the S-axis) are in this model symmetric with respect to the centre of the disc. Hence the picture of any orbit possesses this symmetry. k-symmetric orbits are symmetric with respect to the vertical axis of the disc. A complete representation of an orbit, either asymmetric or k-symmetric, requires two copies of this model, one for the upper and the other for the lower half hyperboloid.

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